SCHWARZ ALTERNATING PROCEDURE FOR ELLIPTIC PROBLEMS DISCRETIZED BY LEAST SQUARES MIXED FINITE ELEMENTS

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Abstract. We consider a two-level domain decomposition method for solving the matrix equations that arise from applying the recently developed least squares mixed finite element method to elliptic boundary value problems. We prove under general conditions that it converges linearly with a factor bounded independent of the number of unknowns.

Key words. domain decomposition, least squares discretization, second–order elliptic problems, iterative methods.

AMS(MOS) subject classifications. 65F10, 65F30

1. Introduction.

We are concerned with solution methods for the matrix equations arising from the least squares mixed finite element method (LSM) applied to second–order elliptic problems, which was recently proposed by Chang [5] and by Pehlivanov, Carey, and Lazarov [11]. The least squares mixed finite element approach is to minimize a least squares residual introduced for the mixed system. Its finite element approximation yields a symmetric and positive definite discrete system and is not subject to the LBB–condition (see [2]). Chang's LSM method differs from that of Pehlivanov et al. in that it incorporates a curl free constrain, which is not neccessary (see [11]) for the approximation accuracy from theoretical point of view, but it is important for efficiency of solution method (see Remark 4.2). Optimal multigrid solver for the system of linear equations resulting in Chang's LSM method for general second–order elliptic boundary value problems will be disscussed in the forthcoming paper.

The objective of this paper is to develop a Schwarz alternating procedure for solving the systems of linear equations resulting from discretizations of the least squares mixed finite element methods and to analyze its convergence. Domain decomposition methods for *standard* finite element and mixed finite element discretizations of second–order elliptic problems have been developed and analyzed in [1], [4], [7], and [9], for example. Here we use a similar domain decomposition approach, but apply it instead to the least squares systems, which are distinguished by the property that they arise from the (least squares) variational principles applied to the first–order systems associated with second–order elliptic problems. We describe the least squares mixed finite element methods in the next section, develop the domain decomposition

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schemes in Section 3, and establish convergence factor bounds in the final section.

2. Least Squares Mixed Finite Element Methods. Consider the following elliptic boundary value problem:

$$\left\{\begin{array}{rcll} -\nabla\cdot\nabla p &=& f, & \text{in} \quad \Omega, \\ \nabla p\cdot\mathbf{n} &=& 0, & \text{on} \quad \Gamma_N, \\ p &=& 0, & \text{on} \quad \Gamma_D, \end{array}\right. \tag{2.1}$$

where Ω is a polygonal domain in R^d ($d = 2$ or 3) with Lipschitz boundary $\Gamma = \Gamma_D \cup \Gamma_N$ such that Γ_D is non-empty. The symbols $\nabla\cdot$ and ∇ stand for the divergence and gradient operators, respectively, and $\mathbf{n}$ denotes the unit outward normal vector on Γ_N. By introducing the flux

$$\mathbf{u} = -\nabla p,$$

problem (2.1) may be rewritten as a system of first–order partial differential equations:

$$\left\{\begin{array}{rcll} \mathbf{u}+\nabla p &=& \mathbf{0}, & \text{in} \quad \Omega, \\ \nabla\cdot\mathbf{u} &=& f, & \text{in} \quad \Omega, \\ \mathbf{u}\cdot\mathbf{n} &=& 0, & \text{on} \quad \Gamma_N, \\ p &=& 0, & \text{on} \quad \Gamma_D. \end{array}\right. \tag{2.2}$$

Let **curl** denote the curl operator. The curl of a two–dimensional vector field is a scalar, but to avoid further notation, where there is no danger of confusion, we nevertheless denote it as a vector. Note that solution, $\mathbf{u}$, of (2.2) is curl free, i.e., $\mathbf{curl}\,\mathbf{u} = \mathbf{0}$, and that homogeneous Dirichlet boundary conditions give $\gamma_\tau\mathbf{u}\,|_{\Gamma_D} = 0$, where $\gamma_\tau\mathbf{u} = \mathbf{u}\cdot\tau$ for $d = 2$ and $\gamma_\tau\mathbf{u} = \mathbf{u}\times\mathbf{n}$ for $d = 3$. Here, τ is the unit tangent to the boundary Γ_D. Then an equivalent extended system for (2.2) is

$$\left\{\begin{array}{rcll} \mathbf{u}+\nabla p &=& \mathbf{0}, & \text{in} \quad \Omega, \\ \nabla\cdot\mathbf{u} &=& f, & \text{in} \quad \Omega, \\ \mathbf{curl}\,\mathbf{u} &=& \mathbf{0}, & \text{in} \quad \Omega, \\ \mathbf{u}\cdot\mathbf{n} &=& 0, & \text{on} \quad \Gamma_N, \\ p &=& 0, & \text{on} \quad \Gamma_D, \\ \gamma_\tau\mathbf{u} &=& 0, & \text{on} \quad \Gamma_D. \end{array}\right. \tag{2.3}$$

Let $(\cdot,\,\cdot)_{0,\Omega}$ denote the inner product in $L^2(\Omega)^d$, $\|\cdot\|_{0,\Omega}$ its induced norm, and $H^m(\Omega)^d$ the standard Sobolev space with norm $\|\cdot\|_{m,\Omega}$ and seminorm $|\cdot|_{i,\Omega}$ $(0 \le i \le m)$. We shall use the following spaces to define the least squares variational problem. Let

$$H(\mathrm{div};\,\Omega) = \{\mathbf{v}\in L^2(\Omega)^d \,|\, \nabla\cdot\mathbf{v}\in L^2(\Omega)\},$$

which is a Hilbert space under the norm

$$\|\mathbf{v}\|_{H(div;\Omega)} \equiv \left(\|\mathbf{v}\|_{0,\Omega}^2 + \|\nabla\cdot\mathbf{v}\|_{0,\Omega}^2\right)^{\frac{1}{2}},$$

and its subspace

$$\mathbf{V}_1 = \{\mathbf{v} \in H(\mathrm{div};\, \Omega),\, |\, \mathbf{v}\cdot\mathbf{n}\,|_{\Gamma_N} = 0\}.$$

We set

$$H(\mathbf{curl};\, \Omega) = \{\mathbf{v} \in L^2(\Omega)^d \,|\, \mathbf{curl}\,\mathbf{v} \in L^2(\Omega)^d\},$$

a Hilbert space with norm

$$\|\mathbf{v}\|_{H(\mathbf{curl};\,\Omega)} \equiv \left(\|\mathbf{v}\|^2_{0,\Omega} + \|\mathbf{curl}\,\mathbf{v}\|^2_{0,\Omega}\right)^{\frac{1}{2}},$$

and its subspace

$$\mathbf{U} = \{\mathbf{v} \in H(\mathbf{curl};\, \Omega),\, |\, \gamma_\tau\,\mathbf{v}\,|_{\Gamma_D} = 0\}.$$

Consider

$$P = \{q \in H^1(\Omega) \,|\, q|_{\Gamma_D} = 0\}$$

equipped with the norm $\|\cdot\|_{1,\Omega}$ or seminorm $|\cdot|_{1,\Omega}$, and set

$$\mathbf{V}_2 = \mathbf{V}_1 \cap \mathbf{U}.$$

Let

$$\mathcal{L}_1 = \begin{pmatrix} \mathbf{v} + \nabla q \\ \nabla\cdot\mathbf{v} \end{pmatrix}, \qquad \mathcal{L}_2 = \begin{pmatrix} \mathbf{v} + \nabla q \\ \nabla\cdot\mathbf{v} \\ \mathbf{curl}\,\mathbf{v} \end{pmatrix},$$

$$\mathbf{F}_1 = \begin{pmatrix} \mathbf{0} \\ f \end{pmatrix}, \quad \text{and} \quad \mathbf{F}_2 = \begin{pmatrix} \mathbf{0} \\ f \\ \mathbf{0} \end{pmatrix}.$$

Fixing $i = 1$ or 2, define the quadratic functional

$$\mathcal{J}_i(\mathbf{v},\, q) = \int_\Omega \langle \mathcal{L}_i - \mathbf{F}_i,\, \mathcal{L}_i - \mathbf{F}_i\rangle\, dx, \quad \forall\,(\mathbf{v},\, q) \in \mathbf{V}_i \times P,$$

where $\langle\cdot,\,\cdot\rangle$ is the Euclidean inner product in R^{2+i}. Then the least squares minimization problem for (2.2) (see [11]) or (2.3) (see [5]) is: find a pair $(\mathbf{u}_i,\, p_i) \in \mathbf{V}_i \times P$ such that

$$\mathcal{J}_i(\mathbf{u}_i,\, p_i) = \inf_{\mathbf{v}\in\mathbf{V}_i,\, q\in P} \mathcal{J}_i(\mathbf{v},\, q).$$

The variational formulation is to find a pair $(\mathbf{u}_i,\, p_i) \in \mathbf{V}_i \times P$ such that

$$a_i(\mathbf{u}_i,\, p_i;\, \mathbf{v},\, q) = (f,\, \nabla\cdot\mathbf{v})_{0,\Omega}, \quad \forall\,(\mathbf{v},\, q) \in \mathbf{V}_i \times P, \tag{2.4}$$

where the bilinear forms $a_i(\cdot;\,\cdot)$ are defined by

$$a_1(\mathbf{u},\, p;\, \mathbf{v},\, q) = \int_\Omega \left((\nabla\cdot\mathbf{u})(\nabla\cdot\mathbf{v}) + (\mathbf{u}+\nabla p)(\mathbf{v}+\nabla q)\right)\, dx$$

and

$$a_2(\mathbf{u},\, p;\, \mathbf{v},\, q) = a_1(\mathbf{u},\, p;\, \mathbf{v},\, q) + \int_\Omega (\mathbf{curl}\,\mathbf{u})(\mathbf{curl}\,\mathbf{v})\, dx.$$

Let $\mathcal{T}^h$ be a regular triangulation of the domain Ω (see [6]) and $\mathbf{V}_i^h \times P^h$ be a finite dimensional–subspace of $\mathbf{V}_i \times P$ associated with the triangulation $\mathcal{T}^h$. Then the least squares mixed finite element discretization of problem (2.1) is to find $(\mathbf{u}_i^h,\, p_i^h) \in \mathbf{V}_i^h \times P^h$ such that

$$a_i(\mathbf{u}_i^h,\, p_i^h;\, \mathbf{v},\, q) = (f,\, \nabla\cdot\mathbf{v})_{0,\Omega}, \quad \forall\,(\mathbf{v},\, q) \in \mathbf{V}_i^h \times P^h. \tag{2.5}$$

3. Schwarz Alternating Procedure. In this section, we first develop some preconditioning bilinear forms that are uniformly equivalent to the bilinear forms $a_i(\cdot;\,\cdot)$ but that decouple the unknowns $\mathbf{u}_i$ and p_i. We then use a Schwarz alternating procedure to solve the resulting preconditioned systems.

Define the bilinear forms $b_i(\cdot;\,\cdot) : \mathbf{V}_i^h \times P^h \longrightarrow R$ by

$$b_1(\mathbf{u},\, p;\, \mathbf{v},\, q) = (\mathbf{u},\, \mathbf{v})_{H(\mathrm{div};\,\Omega)} + (\nabla p,\, \nabla q)_{0,\Omega}$$

and

$$b_2(\mathbf{u},\, p;\, \mathbf{v},\, q) = b_1(\mathbf{u},\, p;\, \mathbf{v},\, q) + (\mathbf{curl}\,\mathbf{u},\, \mathbf{curl}\,\mathbf{v})_{0,\Omega}.$$

It has been shown in [11] that the bilinear form $b_1(\cdot;\,\cdot)$ is uniformly equivalent to $a_1(\cdot;\,\cdot)$ with respect to the mesh size h, i.e., there exist positive constants C_1 and C_2 independent of the mesh size h such that

$$C_1\, b_1(\mathbf{v},\, q;\, \mathbf{v},\, q) \le a_1(\mathbf{v},\, q;\, \mathbf{v},\, q) \le C_2\, b_1(\mathbf{v},\, q;\, \mathbf{v},\, q), \quad \forall\,(\mathbf{v},\, q) \in \mathbf{V}_1^h \times P^h. \tag{3.1}$$

Here and henceforth, we use C with or without scripts to denote a generic positive constant independent of the mesh size h and the number of subdomains J (which will be introduced below). Uniform equivalence of the bilinear forms $a_2(\cdot;\,\cdot)$ and $b_2(\cdot;\,\cdot)$ is a straightforward consequence of (3.1). Thus, we can use $b_i(\cdot;\,\cdot)$ to precondition $a_i(\cdot;\,\cdot)$ so that the number of simple iterations (i.e., Richardson's method) applied to the preconditioned problem that guarantees a desired error reduction is bounded independent of the mesh size h. However, each step of the iteration requires solving a system of linear equations:

$$b_i(\mathbf{u},\, p;\, \mathbf{v},\, q) = g(\mathbf{v},\, q), \quad \forall\,(\mathbf{v},\, q) \in \mathbf{V}_i^h \times P^h, \tag{3.2}$$

where g is the appropriate known right–hand side functional. In this section, we describe a two–level Schwarz alternating procedure for this purpose.

To define the Schwarz alternating procedure for problem (3.2), assume that we have a set of overlapping subdomains $\{\Omega_j\}_{j=1}^J$ such that $\bar{\Omega} = \cup_{j=1}^J \bar{\Omega}_j$. Assume that the maximum number of subdomain overlaps is bounded independent of J and, further, that the boundaries of the Ω_j do not cut through any element in $\mathcal{T}^h$, i.e., they must coincide with boundaries of elements of $\mathcal{T}^h$. Thus, the restriction of $\mathcal{T}^h$ to Ω_j provides a uniform triangulation $\mathcal{T}_j^h$ for Ω_j.

Let $\mathbf{V}_{i,j}^h \times P_j^h$ be the corresponding finite element approximation space associated with the triangulation $\mathcal{T}_j^h$, i.e.,

$$\mathbf{V}_{i,j}^h = \{\mathbf{v} \in \mathbf{V}_i^h \,|\, \gamma_\tau \mathbf{v} = 0, \text{ on } \partial\Omega_j \setminus \partial\Omega\}$$

and

$$P_j^h = \{q \in P^h \,|\, q = 0, \text{ on } \partial\Omega_j \setminus \partial\Omega\}.$$

Define the projection operator $\mathbf{P}_{i,j} : \mathbf{V}_i^h \times P^h \longrightarrow \mathbf{V}_{i,j}^h \times P_j^h$ by

$$b_i(\mathbf{P}_{i,j}(\mathbf{v}, q); \mathbf{w}, p) = b_i(\mathbf{v}, q; \mathbf{w}, p), \quad \forall (\mathbf{w}, q) \in \mathbf{V}_{i,j}^h \times P_j^h.$$

ALGORITHM 3.1 (SCHWARZ ALTERNATING PROCEDURE). *Given an approximation* $(\mathbf{u}_i^l, p_i^l) \in \mathbf{V}_i^h \times P^h$ *to the solution* $(\mathbf{u}^h, p^h)$ *of problem* (2.5), *define the next approximation* $(\mathbf{u}_i^{l+1}, p_i^{l+1}) \in \mathbf{V}_i^h \times P^h$ *as follows:*

1. *Set* $W_0 = (\mathbf{u}_i^l, p_i^l)$.
2. *For* $j = 1, 2, ..., J$ *in turn, define* W_j *by*

$$W_j = W_{j-1} + \omega \mathbf{P}_{i,j}((\mathbf{u}_i^h, p_i^h) - W_{j-1}),$$

where $\omega \in (0, 2)$.

3. *Set* $(\mathbf{u}_i^{l+1}, p_i^{l+1}) = W_J$.

REMARK 3.1. *The action of* $\mathbf{P}_{i,j}$ *on* $(\mathbf{u}_i^h, p_i^h) - W_{j-1}$ *can be computed by solving*

$$\begin{aligned} b_i(\mathbf{P}_{i,j}((\mathbf{u}_i^h, p_i^h) - W_{j-1}); \mathbf{v}, q) &= b_i((\mathbf{u}_i^h, p_i^h) - W_{j-1}; \mathbf{v}, q) \\ &= (f, \nabla \cdot \mathbf{v})_{0,\Omega} - b_i(W_{j-1}; \mathbf{v}, q) \end{aligned} \tag{3.3}$$

for any $(\mathbf{v}, q) \in \mathbf{V}_{i,j}^h \times P_j^h$.

A simple calculation implies that the *error propagation* or *iteration operator* for Algorithm 3.1 is

$$\mathbf{E}_{i,1} = (\mathbf{I} - \omega \mathbf{P}_{i,J})(\mathbf{I} - \omega \mathbf{P}_{i,J-1}) \cdots (\mathbf{I} - \omega \mathbf{P}_{i,1}). \tag{3.4}$$

The set of overlapping subdomains $\{\Omega_j\}_{j=1}^J$ used in Algorithm 3.1 is quite general. However, its convergence rate depends on the size of the overlaps. To avoid this, we introduce an appropriate coarse grid mixed finite element approximation subspace. To this end, assume that we are given a regular coarse triangulation $\mathcal{T}^H = \{K_j\}_{j=1}^J$ and a regular fine triangulation $\mathcal{T}^h$ obtained by partitioning the elements of $\mathcal{T}^H$.

Associated with the coarse triangulation $\mathcal{T}^H$, we construct the set of overlapping subdomains $\{\Omega_j\}_{j=1}^J$ by extending each element $K_j \in \mathcal{T}^H$ to a larger subdomain Ω_j, whose diameter is denoted by H_j, such that $\bar{\Omega} = \cup_{j=1}^J \bar{\Omega}_j$. Assume that the maximum number of subdomain overlaps is bounded independent of J and, further, that the distance between the boundaries ∂K_j and $\partial \Omega_j$ is bounded below by $\zeta_1 H$ and above by $\zeta_2 H$, i.e., there exist constants ζ_1, $\zeta_2 > 0$ such that

$$\zeta_1 H \leq \operatorname{dist}(\partial K_j, \partial \Omega_j) \leq \zeta_2 H, \quad \forall j \in \{1, 2, ..., J\}.$$

Also assume that the boundaries of the Ω_j do not cut through any element in $\mathcal{T}^H$, i.e., they must coincide with boundaries of elements of $\mathcal{T}^h$.

Let $\mathbf{V}_i^H \times P^H$ be the mixed finite element approximation subspace associated with $\mathcal{T}^H$, which will be specified in Section 3. Define the projection operator $\mathbf{P}_{i,0}$: $\mathbf{V}_i^h \times P^h \longrightarrow \mathbf{V}_i^H \times P^H$ by

$$b_i(\mathbf{P}_{i,0}(\mathbf{v}, q); \mathbf{w}, p) = b_i(\mathbf{v}, q; \mathbf{w}, p), \quad \forall (\mathbf{w}, q) \in \mathbf{V}_i^H \times P^H.$$

ALGORITHM 3.2 (TWO-LEVEL SCHWARZ ALTERNATING PROCEDURE). *Given an approximation $(\mathbf{u}_i^l, p_i^l) \in \mathbf{V}_i^h \times P^h$ to the solution $(\mathbf{u}_i^h, p_i^h)$ of problem (3.2), define the next approximation $(\mathbf{u}_i^{l+1}, p_i^{l+1}) \in \mathbf{V}_i^h \times P^h$ as follows:*

1. *Set $W_{-1} = (\mathbf{u}_i^l, p_i^l)$.*
2. *For $j = 0, 1, ..., J$ in turn, define W_j by*

$$W_j = W_{j-1} + \omega \mathbf{P}_{i,j}((\mathbf{u}_i^h, p_i^h) - W_{j-1})$$

where $\omega \in (0, 2)$.

3. *Set $(\mathbf{u}_i^{l+1}, p_i^{l+1}) = W_J$.*

As Algorithm 3.1, the error propogation operator for Algorithm 3.2 is

$$\mathbf{E}_{i,2} = (\mathbf{I} - \omega \mathbf{P}_{i,J})(\mathbf{I} - \omega \mathbf{P}_{i,J-1}) \cdots (\mathbf{I} - \omega \mathbf{P}_{i,1})(\mathbf{I} - \omega \mathbf{P}_{i,0}). \tag{3.5}$$

4. Convergence Analysis. For simplicity, we restrict ourself to the continuous piecewise linear finite element approximation spaces, i.e.,

$$P^h = \{q \in C^0(\Omega) \,|\, q|_K \in P_1(K) \quad \forall K \in \mathcal{T}^h, \text{ and } q = 0 \text{ on } \Gamma_D\}$$

and

$$\mathbf{V}_i^h = \{\mathbf{v} \in C^0(\Omega)^d \,|\, v_l|_K \in P_1(K) \quad \forall K \in \mathcal{T}^h, \text{ and } \mathbf{v} \in \mathbf{V}_i\},$$

where $P_1(K)$ is the space of polynomials of degree at most one. For simplicity, we only consider $d = 2$. Extensions of the subsequent results to $d = 3$ and higher–order finite element approximation spaces are straightforward. Letting $I^h : P \longrightarrow P^h$ denote the standard interpolation operator, and defining interpolation operator $\mathbf{I}_i^h : \mathbf{V}_i \longrightarrow \mathbf{V}^h$ by

$$\mathbf{I}_i^h \mathbf{v} = \begin{pmatrix} I^h v_1 \\ I^h v_2 \end{pmatrix}, \quad \forall \mathbf{v} \in \mathbf{V}_i.$$

LEMMA 4.1. *For any* $(\mathbf{v},\, q) \in \mathbf{V}_1^h \times P^h$*, there exist* $(\mathbf{v}_j,\, q_j) \in \mathbf{V}_{1,j}^h \times P_j^h$ *and a positive constant* C_0 *such that*

$$(\mathbf{v},\, q) = \sum_{j=1}^{J} (\mathbf{v}_j,\, q_j) \tag{4.1}$$

and

$$\sum_{j=1}^{J} b_1(\mathbf{v}_j,\, q_j;\, \mathbf{v}_j,\, q_j) \le \frac{C_0}{\delta^2} b_1(\mathbf{v},\, q;\, \mathbf{v},\, q), \tag{4.2}$$

where $\delta > 0$ *is a parameter reflecting the relative size of the overlap of the subdomains (e.g.,* $\delta = 1$ *signifies maximal overlap,* $\delta = 0$ *signifies no overlap).*

Proof. Let $\theta_j(x) \ge 0$ $(j = 1, 2, ..., J)$ be a partition of unity with respect to the set of overlapping subdomains $\{\Omega_j\}_{j=1}^J$ such that

$$\theta_j(x) \in W^{1,\infty}(\Omega) \quad \text{and} \quad |\nabla \theta_j| \le \frac{C}{\delta}.$$

Choosing

$$\mathbf{v}_j = \mathbf{I}_1^h(\theta_j \mathbf{v}) \quad \text{and} \quad q_j = I^h(\theta_j q),$$

then the decomposition in (4.1) follows. It has been shown in [8] that

$$(\nabla\, q_j,\, \nabla\, q_j)_{0,\Omega} \le C\, |\nabla\, q|_{1,\Omega_j}^2 + \frac{C}{\delta^2}\, |q|_{0,\Omega_j}^2. \tag{4.3}$$

In order to estimate the $H(div;\, \Omega)$ norm of $\mathbf{v}_j$, we follow the proof in [8]. For any $K \in \mathcal{T}_h$ and $K \subset \Omega_j$, let $\bar{\theta}_j$ be the average value of θ_j over K. By using the inverse inequality, we have

$$\begin{aligned} |\mathbf{v}_j|_{H(div;K)}^2 &\le 2|\bar{\theta}_j \mathbf{v}|_{H(div;K)}^2 + 2|\mathbf{I}_1^h\, \left((\theta_j - \bar{\theta}_j)\mathbf{v}\right)|_{H(div;K)}^2 \\ &\le C\, |\mathbf{v}|_{H(div;K)}^2 + C\, h^{-2} |\mathbf{I}_1^h\, \left((\theta_j - \bar{\theta}_j)\mathbf{v}\right)|_{0,K}^2 \\ &\le C\, |\mathbf{v}|_{H(div;K)}^2 + \frac{C}{\delta^2} |\mathbf{v}|_{0,K}^2. \end{aligned}$$

The last inequality follows from the facts that the L^2-norm is equivalent to the l_2-norm over K and that θ_j differs from its average over K by at most $C\,\frac{h}{\delta}$. After summing over all elements of Ω_j, we obtain

$$|\mathbf{v}_j|_{H(div;\Omega)}^2 \le |\mathbf{v}_j|_{H(div;\Omega_j)}^2 \le C\, |\mathbf{v}|_{H(div;\Omega_j)}^2 + \frac{C}{\delta^2} |\mathbf{v}|_{0,\Omega_j}^2,$$

which, together with inequality (4.3), implies that

$$b_1(\mathbf{v}_j,\, q_j;\, \mathbf{v}_j,\, q_j) \le C\, b_{1,\Omega_j}(\mathbf{v},\, q;\, \mathbf{v},\, q) + \frac{C}{\delta^2} \left((\mathbf{v},\, \mathbf{v})_{0,\Omega_j} + |q|_{0,\Omega_j}^2\right).$$

Here, $b_{1,\Omega_j}(\mathbf{v},\, q;\, \mathbf{v},\, q) = (\mathbf{v},\, \mathbf{v})_{0,\Omega_j} + (\nabla\cdot\mathbf{v},\, \nabla\cdot\mathbf{v})_{0,\Omega_j} + (A\nabla q_j,\, A\nabla q_j)_{0,\Omega_j}$. (4.2) now follows from summing the above inequality over j and noting the fact that the maximum number of subdomain overlaps is bounded independent of J, i.e.,

$$\sum_{j=1}^{J} b_1(\mathbf{v}_j,\, q_j;\, \mathbf{v}_j,\, q_j) \le C\, b_1(\mathbf{v},\, q;\, \mathbf{v},\, q) + \frac{C_0}{\delta^2}\left(|\mathbf{v}|^2_{0,\Omega} + |q|^2_{0,\Omega}\right) \le \frac{C_0}{\delta^2} b_1(\mathbf{v},\, q;\, \mathbf{v},\, q).$$

This completes the proof. □

LEMMA 4.2. *For any $(\mathbf{v},\, q) \in \mathbf{V}_2^h \times P^h$, there exist $(\mathbf{v}_j,\, q_j) \in \mathbf{V}_{1,j}^h \times P_j^h$ and a positive constant C_0 such that*

$$(\mathbf{v},\, q) = \sum_{j=1}^{J} (\mathbf{v}_j,\, q_j) \tag{4.4}$$

and

$$\sum_{j=1}^{J} b_2(\mathbf{v}_j,\, q_j;\, \mathbf{v}_j,\, q_j) \le \frac{C_0}{\delta^2} b_2(\mathbf{v},\, q;\, \mathbf{v},\, q). \tag{4.5}$$

Proof. Let $\theta_j(x) \ge 0$ $(j = 1, 2, ..., J)$ be the same partition of unity as in Lemma 4.1, choosing

$$\mathbf{v}_j = \mathbf{I}_2^h(\theta_j \mathbf{v}) \quad \text{and} \quad q_j = I^h(\theta_j q),$$

then the decomposition (4.4) follows. The same argument in Lemma 4.1 gives

$$(\mathbf{curl}\, \mathbf{v}_j,\, \mathbf{curl}\, \mathbf{v}_j)_{0,\Omega_j} \le C\, (\mathbf{curl}\, \mathbf{v},\, \mathbf{curl}\, \mathbf{v})_{0,\Omega_j} + \frac{C}{\delta^2}(\mathbf{v},\, \mathbf{v})_{0,\Omega_j}.$$

By the inequalities in the proof of Lemma 4.1, we have

$$b_2(\mathbf{v}_j,\, q_j;\, \mathbf{v}_j,\, q_j) \le C\, b_{2,\Omega_j}(\mathbf{v},\, q;\, \mathbf{v},\, q) + \frac{C}{\delta^2}\left((\mathbf{v},\, \mathbf{v})_{0,\Omega_j} + |q|^2_{0,\Omega_j}\right).$$

An argument analogous to that in the proof of Lemma 4.1 leads to inequality (4.5). □

Let $|||\cdot|||_i$ denote the norm induced by the bilinear form $b_i(\cdot;\, \cdot)$.

THEOREM 4.1. *The convergence factors of* Algorithm 3.1 *are bounded by $\gamma_{i,1} \equiv 1 - C_{i,1}\,\delta^2 < 1$, i.e.,*

$$|||\mathbf{E}_{i,1}|||_i \le \gamma_{i,1} < 1. \tag{4.6}$$

Proof. We will illustrate the proof for $i = 1$ only since the analysis for $i = 2$ is similar. For any $(\mathbf{v},\, q) \in \mathbf{V}_1^h \times P^h$, by Lemma 4.1, we have

$$b_1(\mathbf{v},\, q;\, \mathbf{v},\, q) \;=\; C \sum_{j=1}^{J} b_1(\mathbf{P}_{1,j}(\mathbf{v},\, q);\, \mathbf{v}_j,\, q_j)$$

$$
\begin{aligned}
&\le C \sum_{j=1}^{J} b_{1,\Omega_j}^{\frac{1}{2}}(\mathbf{P}_{1,j}(\mathbf{v}, q); \mathbf{P}_{1,j}(\mathbf{v}, q)) b_{1,\Omega_j}^{\frac{1}{2}}(\mathbf{v}_j, q_j; \mathbf{v}_j, q_j) \\
&\le C \left(\sum_{j=1}^{J} b_{1,\Omega_j}(\mathbf{P}_{1,j}(\mathbf{v}, q); \mathbf{P}_{1,j}(\mathbf{v}, q)) \right)^{\frac{1}{2}} \left(\sum_{j=1}^{J} b_1(\mathbf{v}_j, q_j; \mathbf{v}_j, q_j) \right)^{\frac{1}{2}} \\
&\le \frac{C}{\delta} \left(\sum_{j=1}^{J} b_1(\mathbf{P}_{1,j}(\mathbf{v}, q); \mathbf{P}_{1,j}(\mathbf{v}, q)) \right)^{\frac{1}{2}} b_1^{\frac{1}{2}}(\mathbf{v}, q; \mathbf{v}, q),
\end{aligned}
$$

which implies that

$$
b_1(\mathbf{v}, q; \mathbf{v}, q) \le \frac{C}{\delta^2 \omega^2} \sum_{j=1}^{J} b_1(\omega \mathbf{P}_{1,j}(\mathbf{v}, q); \omega \mathbf{P}_{1,j}(\mathbf{v}, q)).
$$

By definition of the projection operators, we also have that

$$
b_1(\omega \mathbf{P}_{1,j}(\mathbf{v}, q); \omega \mathbf{P}_{1,j}(\mathbf{v}, q)) = \omega b_1(\omega \mathbf{P}_{1,j}(\mathbf{v}, q); \mathbf{v}, q)
$$

with $\omega \in (0, 2)$. Hence, inequality (4.6) follows from Theorem 2.1 in [3], which completes the proof. □

Assume that we are given mixed finite element approximation subspace $\mathbf{V}_i^H \times P^H$ associated with the coarse triangulation $\mathcal{T}^H$. Let $\mathbf{Q}_i^H : \mathbf{V}_i^h \longrightarrow \mathbf{V}_i^H$ and $Q^H : P^h \longrightarrow P^H$ be two operators that satisfy

$$
\|q - Q^H q\|_0 \le C\, H |q|_{1,\Omega} \quad \text{and} \quad \|Q^H q\|_{1,\Omega} \le C\, |q|_{1,\Omega}, \quad \forall\, q \in P^h, \tag{4.7}
$$

and

$$
\|\mathbf{v} - \mathbf{Q}_i^H \mathbf{v}\|_{0,\Omega} \le C\, H\, |||\mathbf{v}|||_i \quad \text{and} \quad |||\mathbf{Q}_i^H \mathbf{v}|||_i \le C\, |||\mathbf{v}|||_i, \quad \forall\, \mathbf{v} \in \mathbf{V}_i^h. \tag{4.8}
$$

Lemma 4.3. *Assume that inequalities* (4.7) *and* (4.8) *for* $i = 1$ *are valid. Then for any* $(\mathbf{v}, q) \in \mathbf{V}_1^h \times P^h$, *there exist* $(\mathbf{v}_j, q_j) \in \mathbf{V}_{1,j}^h \times P_j^h$ *and a positive constant* C_0 *such that*

$$
(\mathbf{v}, q) = \sum_{j=0}^{J} (\mathbf{v}_j, q_j) \tag{4.9}
$$

and

$$
\sum_{j=0}^{J} b_1(\mathbf{v}_j, q_j; \mathbf{v}_j, q_j) \le C\, b_1(\mathbf{v}, q; \mathbf{v}, q). \tag{4.10}
$$

Proof. For any $(\mathbf{v}, q) \in \mathbf{V}_1^h \times P^h$, let $\mathbf{w} = \mathbf{v} - \mathbf{Q}_1^H \mathbf{v}$ and $p = q - Q^H q$. Then we have a decomposition of $(\mathbf{w}, p)$ in $\mathbf{V}_{1,j}^h \times P_j^h$ $(j = 1, 2, ..., J)$ similar to that in the proof of Lemma 4.1, i.e.,

$$
\mathbf{v}_j = \mathbf{I}_1^h(\theta_j \mathbf{w}) \quad \text{and} \quad q_j = I^h(\theta_j p)
$$

and they satisfy

$$\sum_{j=1}^{J} b_1(\mathbf{v}_j,\, q_j;\, \mathbf{v}_j,\, q_j) \le C\, b_1(\mathbf{w},\, p;\, \mathbf{w},\, p) + \frac{C_0}{H^2}\left(|\mathbf{w}|_{0,\Omega}^2 + |p|_{0,\Omega}^2\right).$$

Hence, by (4.8) and (4.9), we have

$$\begin{aligned}\sum_{j=0}^{J} b_1(\mathbf{v}_j,\, q_j;\, \mathbf{v}_j,\, q_j) &= b_1(\mathbf{Q}_1^H\mathbf{v},\, Q^H q;\, \mathbf{Q}_1^H,\, Q^H q) + \sum_{j=1}^{J} b_1(\mathbf{v}_j,\, q_j;\, \mathbf{v}_j,\, q_j)\\ &\le b_1(\mathbf{v},\, q;\, \mathbf{v},\, q) + \frac{C_0}{H^2}\left(|\mathbf{w}|_{0,\Omega}^2 + |p|_{0,\Omega}^2\right)\\ &\le C\, b_1(\mathbf{v},\, q;\, \mathbf{v},\, q).\end{aligned}$$

This completes the proof. □

LEMMA 4.4. *Assume that inequalities* (4.7) *and* (4.8) *for* $i = 2$ *are valid. Then for any* $(\mathbf{v},\, q) \in \mathbf{V}_2^h \times P^h$*, there exist* $(\mathbf{v}_j,\, q_j) \in \mathbf{V}_{2,j}^h \times P_j^h$ *and a positive constant* C_0 *such that*

$$(\mathbf{v},\, q) = \sum_{j=0}^{J} (\mathbf{v}_j,\, q_j) \tag{4.11}$$

and

$$\sum_{j=0}^{J} b_2(\mathbf{v}_j,\, q_j;\, \mathbf{v}_j,\, q_j) \le C\, b_2(\mathbf{v},\, q;\, \mathbf{v},\, q). \tag{4.12}$$

Proof. The proof is analogous to that of Lemma 4.3. □

According to Remark 2.2 in [3], by using Lemmas 4.3 and 4.4, an argument analogous to the proof of Theorem 4.1 gives the following convergence estimate for Algorithm 3.2. (Note that the constants $\gamma_{i,2}$ do not depend on δ^2 because, unlike (4.2) and (4.5), inequalities (4.10) and (4.12) do not depend on δ^2.)

THEOREM 4.2. *Under the assumptions of* Lemma 4.3*, the convergence factors of* Algorithm 3.2 *are bounded by constants* $\gamma_{i,2} < 1$ *independent of* δ^2*, i.e.,*

$$|||\mathbf{E}_{i,2}|||_i \le \gamma_{i,2} < 1. \tag{4.13}$$

REMARK 4.1. *For* $i = 2$*, let* $\mathbf{V}_2^H \times P^H$ *be the continuous piecewise linear finite element space associated with the coarse triangulation* $\mathcal{T}^H$*. Then it is well known that inequalities* (4.7) *holds for the standard* L^2 *projection operator. Because the* $|||\cdot|||_2$ *norm is equivalent to the* $H^1(\Omega)^d$ *norm (see [10]), it is also known that* (4.8) *is valid for the standard* L^2 *projection operator.*

REMARK 4.2. *For* $i = 1$*, unfortunately,* (4.8) *is not valid for the continuous piecewise linear finite element space associated with the coarse triangulation* $\mathcal{T}^H$*. To* *overcome this difficulty, we could choose $P^H \subset P^h$ to be a finite element approximation subspace in $P \cap H^2(\Omega)$ associated with $\mathcal{T}^H$ that satisfies inequality (4.7) for the standard L^2 projection operator, and define $\mathbf{V}_1^H = \nabla P^H$. It is now easy to check the validity of inequality (4.8) for the standard L^2 projection operator. But this expresses a major disadvantage of the least squares mixed finite element method posed without the curl free constraint, because many of the best known and widely used finite element spaces are excluded.*

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REFERENCES

[1] J. H. Bramble, J. E. Pasciak, J. Wang, and J. Xu, *Convergence estimates for product iterative methods with applications to domain decomposition and multigrid*, Math. Comp., 57(1991), pp. 1–21.

[2] F. Brezzi, *On the existence, uniqueness and approximation of saddle point problems arising from Lagrange multipliers*, RAIRO Ser. Anal. Numer., 8(1974), no. R–2, pp. 129–151.

[3] Z. Cai, *Norm estimates of product operators with application to domain decomposition*, Appl. Math. Comp., 53(1993), pp. 251–276.

[4] Z. Cai and X. Ye, *Domain decomposition for mixed finite element method in 3–D*, manuscript.

[5] C. L. Chang, *Finite element approximation for grad-div type systems in the plane*, SIAM J. Numer. Anal., 29(1992), pp.452-461.

[6] P. G. Ciarlet, *The Finite Element Method for Elliptic Problems*, North-Holland, New York, 1978.

[7] M. Dryja and O. Widlund, *An additive variant of the Schwarz alternating method for the case of many subregions*, Tech. Rep. 339, Courant Institute, 1987.

[8] M. Dryja and O. Widlund, *Some domain decomposition algorithms for elliptic problems*, in Iterative Methods for Large Linear Systems, San Diego, California, 1989, Academic Press, pp. 273–291.

[9] R. E. Ewing and J. Wang, *Analysis of the Schwarz algorithm for mixed finite element methods*, RAIRO Math. Modelling Numer. Anal., 26(1992), pp. 739–756.

[10] V. Girault and P. A. Raviart, *Finite Element Methods for Navier-Stokes Equations: Theory and Algorithms*, Springer-Verlag, New York, 1986.

[11] A. I. Pehlivanov, G. F. Carey, and R. D. Lazarov, *Least squares mixed finite elements for second order elliptic problems*, manuscript.

[12] X. Zhang, *Studies in Domain Decompostion: Multilevel Methods and Biharmonic Dirichlet Problem*, Ph.D thesis, Courant Institute, New York University, September, 1991.