Multilevel Adaptive Methods for Elliptic Eigenproblems: A Two-Level Convergence Theory[1]

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Abstract. The fast adaptive composite grid method (FAC) is a multilevel scheme that attempts to provide accurate and optimal computation of solutions of partial differential equations on refined grids. The central feature of FAC that ensures its efficiency is its *domain-decomposition-like* use of uniform grids that cover different regions with different meshsizes, but that fully overlap. This full overlap is geometric in the sense that the regions covered by the finer grids are also covered by the coarser ones, but *only at the coarser grid scales.* FAC's full overlap yields fast convergence rates, while its limited scales in the overlap region assure optimal complexity.

This paper develops a two-level convergence theory for FAC applied to elliptic eigenproblems. The FAC method is realized by constructing a refined grid version of RQMG, which is multilevel Rayleigh quotient minimization scheme developed in an earlier paper. The theory shows that this form of FAC converges independent of the mesh sizes of either the global or local grids. As such, this is the first theory established for multilevel adaptive methods beyond linear equations.

Introduction

The fast adaptive composite grid method (FAC) is a multilevel scheme designed for efficient and accurate discretization and solution of partial differential equations on refined grids. FAC is based on the use of a *composite* grid, which is the union of two or more uniform grids covering different regions and using different mesh sizes. See the example in Figure 1, which consists of one global level with four nested refinements in a single refinement region. The optimal performance of FAC observed in many practical situations stems from its use of the constituent uniform grids that, in a sense, precondition

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the composite grid equations. See [McCormick 1989] and the references cited therein for a treatment of FAC, including numerical results for various types of equations (e.g., elliptic, parabolic, and hyperbolic) and theory for the linear case.

FAC was introduced in [McCormick 1984] and [McCormick and Thomas 1986]. It has some of the same structure as the multilevel adaptive methods (MLAT; cf. [Brandt 1973, 1977]). However, the basic approaches are quite different: MLAT generalizes multigrid methods from the uniform to the refined grid case by a convenient but occasionally tricky premise that finer grids need not cover the entire domain; FAC instead focuses systematically on the composite grid and develops a multilevel solver for its equations. A technique quite similar to FAC was developed in [Bramble, Ewing, Pasciak, and Schatz 1988]; the two are compared in [Ewing, Lazarov, and Vassilevski 1988].

The purpose of this paper is to develop a simple theory for FAC applied to elliptic eigenproblems. To obtain this simplicity, focus will be on the algebraic eigenproblem associated with a two-level composite grid as depicted in Figure 2. Assumptions will be made that correspond to self-adjoint elliptic eigenvalue problems with homogeneous Dirichlet boundary conditions discretized by finite elements, but the theory has rather more general applicability. In any case, explicit reference to the infinite-dimensional setting will be made only in the final section. Also, this paper analyzes only the basic scheme that uses exact uniform grid solvers and fine-to-coarse cycling. Other two-level schemes that use approximate uniform grid solvers and nested iteration, for example, can be treated in a manner entirely analogous to the linear case (c.f. [McCormick 1989]). However, a special challenge would be to extend the sophisticated multilevel FAC theory of [Widlund 1989]; this would likely require a different approach from the one used here.

It should be reemphasized that, while the theory is for exact uniform grid solvers, it can be applied to establish optimal complexity of practical schemes that use approximate solvers, provided they are optimal themselves. For instance, the RQMG scheme studied numerically in [Mandel and McCormick 1989] uses a multigrid technique for the global coarse-grid problem that is compatible with the sub-space minimization philosophy taken here, and nonlinear Gauss-Seidel for the local fine-grid problem. Theory for mesh-independent convergence and, therefore, optimal complexity of the global-grid scheme is developed in [Cai, Mandel, and McCormick 1992]. The point of the theory developed here is that FAC complexity is completely determined by the complexity of the subgrid solvers, and any suitable solver could be considered for this purpose.

In the next section, to improve perspective, FAC is interpreted as a domain-decomposition-like method. Section 3 develops the notation, assumptions, and basic algorithm for FAC applied to the composite grid eigenproblem, and Section 4 analyzes convergence of the basic algorithm. The theory is applied to a class of elliptic equations in Section 5.

2. FAC Domain Decomposition

A general algebraic problem defined on the composite grid exemplified by Figure

2 can be treated by a *classical* domain decomposition multiplicative Schwarz-like process. This would typically involve choosing an initial guess that satisfies the boundary conditions, then iterating back and forth between solvers on the *fine grid (patch)* and *outer coarse grid (L-shaped)*. In a sense, this means that the composite grid operator is being preconditioned by the product of the imbedded inverse of the operators defined on the patch and L-shaped regions.

Unfortunately, this classical approach is usually suboptimal: its convergence rate degrades with increasing number of points in the *interface* (i.e., the coarse grid points at the artificial patch boundary). In fact, for the multilevel case, this rate also degrades with increasing number of levels. A cure to improve this degradation would be to allow domain overlap, but this would generally be too costly, and it would defeat the purpose of restricting computation to uniform grids. FAC avoids these difficulties by using a coarse grid that extends fully under the patch. This *scale-limited full domain overlap* is what gives FAC its optimal overall complexity. Note that FAC can thus be interpreted as a method that preconditions the composite grid operator by the product of the imbedded inverses of the fine grid patch and *global coarse grid* operators.

3. Preliminaries

This section introduces the notation and basic algorithm and makes certain assumptions necessary for the general theory of Section 4. The target generalized eigenproblem is defined on the real finite-dimensional *composite grid space* $\mathcal{H}$ ($\mathcal{H}$ is thought to contain vectors whose entries correspond to nodal values on the composite grid):

$$Au = \lambda Bu, \qquad u \neq 0. \tag{1}$$

Here, $A, B : \mathcal{H} \to \mathcal{H}$ are self-adjoint linear operators in the inner product $\langle \cdot, \cdot \rangle$ and B is positive definite. Such a vector u will be called an eigenvector of (A, B) belonging to λ. The notation $(\cdot, \cdot)$ will denote the *B-inner product:*

$$(v, w) = \langle v, Bw \rangle, \qquad v, w, \in \mathcal{H}.$$

The norm induced by $\langle \cdot, \cdot \rangle$ is denoted by $\| \cdot \|$; the *B-norm* induced by $(\cdot, \cdot)$ is denoted by $\| \cdot \|_B$. If C is a nonnegative definite self-adjoint matrix, then two vectors v and w are called *C-orthogonal* if $\langle v, Cw \rangle = 0$. Thus, B-orthogonality means that $(v, w) = 0$. Similarly, a *C-unit* vector v is one for which $\langle v, Cv \rangle = 1$.

For simplicity of analysis, it is henceforth assumed that the smallest eigenvalue for (1) is $\lambda = 0$. (The more general case, where one end of the spectrum for (A, B) is bounded, can be analyzed by transforming (1) with a shift and possible sign change: $(\pm A - \rho B)u = (\pm \lambda - \rho)Bu$.) It is also assumed that $\lambda = 0$ is simple. Let u denote its (unique up-to-sign) eigenvector of B-unit length: $\| u \|_B = 1$.

Let the Rayleigh quotient RQ be defined by

$$\mathrm{RQ}(v) = \frac{\langle v, Av \rangle}{\langle v, Bv \rangle}, \qquad v \in \mathcal{H} - \{0\}.$$

Then finding the smallest eigenvalue and its B-unit eigenvector for (1) is equivalent to finding the B-unit vector (unique up-to-sign) $u \in \mathcal{H}$ that satisfies

$$\mathrm{RQ}(u) = \min \mathrm{RQ}(v). \tag{2}$$

The eigenvalue is given by $\lambda = \mathrm{RQ}(u)$ ($= 0$ by assumption).

A directional descent method for solving (2) starts with an initial guess $v \in \mathcal{H} - \{0\}$, then chooses a direction $d \in \mathcal{H} - \{0\}$ and sets

$$v \leftarrow v - sd,$$

where s is chosen to minimize $\mathrm{RQ}(v - sd)$ over all possible $s \in \mathcal{R}$. Since $\mathrm{RQ}(v - sd)$ is a rational function of quadratics in s, its minimum can be computed directly and efficiently. However, the major issue with this approach is proper choice of direction d. Conventional schemes (e.g., conjugate directions, coordinate relaxation, and steepest descent) are generally slow to converge because they cannot effectively reduce smooth error components. The multilevel scheme RQMG was developed in [Mandel and McCormick 1989] to overcome this slowness. *The objective of this paper is to provide a theoretical foundation for the two-level composite grid version of RQMG applied to (1).*

A practical description of this version of RQMG is given in [Mandel and McCormick 1989] together with sample numerical results. Here, only an abstract formulation is presented. To this end, the following additional constructs are developed. Let the *coarse grid space* (associated with the global coarse subgrid of the composite grid) be represented by G and let the *fine grid space* (associated with the local refinement patch) be represented by P. Let $\langle \cdot, \cdot \rangle$ be an appropriate inner product and $\| \cdot \|$ its induced norm for G, which will be distinguishable from that for $\mathcal{H}$ by context. Let $I : G \to \mathcal{H}$ and $J : P \to \mathcal{H}$ be interpolation operators that satisfy the following conditions:

- I is linear and of full rank, $\| I^T r \| = \| r \|$ for any $r \in \mathcal{H}$ that is zero on the patch, and $\| I \| = 1$.
- J is the imbedding operator defined so that $p = Jv$ agrees with v on the patch interior and is zero elsewhere. (Note that J is therefore also linear and of full rank.)

These conditions are naturally satisfied by interpolation operators used in standard applications (cf. [McCormick 1989]).

One *coarse-to-fine two-grid exact-subgrid-solver* cycle of RQMG applied to $v \in \mathcal{H} - \{0\}$ is represented by the expression

$$v \leftarrow \mathrm{RQMG}(v).$$

It is described formally by the following two steps:

(i) Compute a $p \in P$ that minimizes $\mathrm{RQ}(v - Jp)$ as a function of p; set $v \leftarrow v - Jp$.

(ii) Compute a $g \in G$ that minimizes $\mathrm{RQ}(v - Ig)$ as a function of g; set $v \leftarrow v - Ig$.

To simplify the following discussion, define the respective ranges of I and J by $\mathcal{G} = IG$ and $\mathcal{P} = JP$, which are subspaces of $\mathcal{H}$. The two steps of RQMG then read as follows:

(i) Compute a $p \in \mathcal{P}$ that minimizes $\mathrm{RQ}(v - p)$ as a function of p; set $v \leftarrow v - p$.

(ii) compute a $g \in \mathcal{G}$ that minimizes $\mathrm{RQ}(v - g)$ as a function of g; set $v \leftarrow v - g$.

We emphasize that this formal description of RQMG may not be defined for all v. For example, the minimum in (i) may not be attained if the minimizer of $\mathrm{RQ}(w)$ over w in the subspace $\{v\}^\infty + \mathcal{P}$ is actually in $\mathcal{P}$. This subspace formulation is a way to pose (i) to avoid this technical difficulty, but it has less of the appearance of a conventional relaxation scheme. In any case, we avoid this technicality locally (i.e., near u where our theory applied) by henceforth assuming that u is not contained in either $\mathcal{G}$ or $\mathcal{P}$. (Otherwise, RQMG converges in one step.)

To analyze RQMG, use will be made of the subspaces $\mathcal{H}_\perp = \{w \in \mathcal{H} : (w, u) = 0\}$ and $\mathcal{Y} = \{y \in \mathcal{H}_\perp : \langle Ay, p\rangle = 0 \text{ for all } p \in \mathcal{P}\}$, and the *A-seminorm* $|||\cdot|||$ induced by the *A-bilinear form*:

$$|||v||| = \langle v, Av\rangle^{1/2}.$$

The theory here will use two different decompositions of $\mathcal{H}$. The first is B-orthogonal:

$$\mathcal{H} = \{u\}^\infty \oplus \mathcal{H}_\perp. \tag{3}$$

The second decomposition is formed first by observing that $\{u\}^\infty + \mathcal{P}$ is an A-orthogonal complement of $\mathcal{Y}$ in $\mathcal{H}$. Therefore, if $u \notin \mathcal{P}$, which is henceforth assumed, then the following is an A-orthogonal decomposition of $\mathcal{H}$:

$$\mathcal{H} = \{u\}^\infty \oplus \mathcal{Y} \oplus \mathcal{P}. \tag{4}$$

Finally, the following central assumptions are made:

Constants $\eta \in (0, 1]$ and $\theta \in [1, \infty)$ exist so that, for each $y \in \mathcal{Y}$, there exists a g in $\mathcal{G}$ such that

$$|||y - g|||^2 \leq (1 - \eta)|||y|||^2 \tag{5}$$

and

$$||g||_B \leq \theta ||g_\perp||_B, \tag{6}$$

where subscript $\perp$ denotes the B-orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_\perp$. Moreover, there exists a constant $\zeta \in [1, \infty)$ such that

$$||p||_B \leq \zeta ||p_\perp||_B \qquad \text{for all } p \in \mathcal{P}. \tag{7}$$

REMARK. *(5) is equivalent to the assertion that a two-level fine-to-coarse FAC cycle applied to the linear problem $Ay = 0$ converges linearly in energy on $\mathcal{H}_\perp$ with factor $\sqrt{1-\eta}$. This equivalence can be seen by noting that the first step of FAC leaves $y \in Y$, and the second step determines a $g \in \mathcal{G}$ that minimizes $|||y - g|||$. (6) asserts that the global-grid function used to approximate y, which is orthogonal to u, does not have a dominant u component. Finally, (7) states that a function in $\mathcal{P}$, which has local support, also cannot have a dominant u component. Theorem 2 will establish these inequalities in terms of several quantities related to the composite grid eigenproblem and its patch and global grid approximations. This general theory will then be applied in Section 5 to a class of self-adjoint elliptic eigenproblems, using the property that A is positive definite on $\mathcal{H}_\perp$:*

$$\langle w, Aw \rangle \geq \gamma(w, w) \qquad \text{for all } w \in \mathcal{H}_\perp.$$

Note that this property can be written differently as

$$|||w||| \geq \gamma^{1/2} ||w||_B \qquad \text{for all } w \in \mathcal{H}_\perp. \tag{8}$$

4. General Theory

The following theorem shows that RQMG converges linearly, as measured by the error in the Rayleigh quotient, with a factor bounded asymptotically by $1 - \eta$. For this purpose, for any $\tau > 0$, define

$$\eta_\tau = \sqrt{\eta} - (1 + \sqrt{\eta})\zeta\tau$$

and

$$\gamma_\tau = \frac{1 - \eta_\tau^2}{(1 - \theta\eta_\tau\tau)^2}.$$

THEOREM 1. *Let $v_1 \in \mathcal{H} - \{0\}$ and $v_2 = RQMG(v_1)$. Let $\varepsilon_i = RQ(v_i) = RQ(v_i) - \lambda$, which is the error in the Rayleigh quotient, $i = 1, 2$. Then, for sufficiently small ε_1, RQMG converges linearly according to the inequality*

$$\varepsilon_2 \leq \gamma_\tau \varepsilon_1, \tag{9}$$

where $\tau = \sqrt{\varepsilon_1/\gamma}$.

PROOF: Let v be the approximation after Step 1 of RQMG applied to v_1. Then

$$\varepsilon \equiv \mathrm{RQ}(v) \leq \mathrm{RQ}(v_1) = \varepsilon_1.$$

Moreover, for sufficiently small τ (i.e., for sufficiently small ε_1), γ_τ decreases with decreasing τ, so it may be assumed that $\tau = \sqrt{\varepsilon/\gamma}$, i.e., $\varepsilon = \varepsilon_1$. Since v results from an optimal correction from $\mathcal{P}$, it follows that

$$\langle v, (A - \varepsilon B)p \rangle = 0 \qquad \text{for all } p \in \mathcal{P}. \tag{10}$$

Consider the expansion guaranteed by (3):

$$v = u + w, \qquad w \in \mathcal{H}_\perp. \tag{11}$$

(The scale of v is unimportant; since $\lambda = 0$ is isolated and $\mathrm{RQ}(v_1) - \lambda$ is sufficiently small, then the coefficient of u in an expansion of v must be nonzero; it may therefore be assumed that this coefficient is 1, as is done in (11).) Consider also the expansion guaranteed by (4):

$$v = \alpha u + y + p_0, \qquad y \in \mathcal{Y}, \qquad p_0 \in \mathcal{P}. \tag{12}$$

(Because p_0 is generally not B-orthogonal to u, it may *not* be assumed here that $\alpha = 1$.) Then (10) and (11) imply that

$$\langle v, Ap\rangle = \varepsilon(u + w, p) \qquad \text{for all } p \in \mathcal{P}.$$

Choosing $p = p_0$ and using (12), then

$$\begin{aligned} |||p_0|||^2 &= \langle p_0, Ap_0\rangle \\ &= \varepsilon(u + w, p_0) \\ &\leq \varepsilon \parallel u + w \parallel_B \cdot \parallel p_0 \parallel_B \\ &\leq \zeta\tau|||w||| \cdot |||p_0|||. \end{aligned}$$

The last inequality follows from (7), (8), the definition of ε, and the fact that $|||p_{0\perp}||| = | \parallel p_0||||$. Hence,

$$|||p_0|||| \leq \zeta\tau|||w|||. \tag{13}$$

Now (5) implies that

$$\begin{aligned} \langle y, Ag\rangle &\geq \frac{1}{2}(|||y|||^2 + |||g|||^2 - (1-\eta)|||y|||^2) \\ &= \frac{1}{2}(\eta|||y|||^2 + |||g|||^2) \\ &\geq \sqrt{\eta}|||y||| \cdot |||g|||. \end{aligned}$$

¿From this, (11)-(13), and the inequality

$$|||w||| \leq |||y||| + |||p_0|||,$$

it follows that

$$\begin{aligned} \langle w, Ag\rangle &= \langle y + p_0, Ag\rangle \\ &\geq (\sqrt{\eta}|||y||| - |||p_0|||)|||g||| \\ &\geq (\sqrt{\eta}|||w||| - (1+\sqrt{\eta})|||p_0|||)|||g||| \\ &\geq \eta_\tau|||w||| \cdot |||g|||. \end{aligned}$$

Hence, setting $s = \eta_\tau \frac{|||w|||}{|||g|||}$, then this inequality and (11) imply that

$$\begin{aligned} \mathrm{RQ}(v - sq) &= \frac{\langle v, Av\rangle - 2s\langle v, Ag\rangle + s^2\langle g, Ag\rangle}{\| v - sg \|_B^2} \\ &= \frac{\langle w, Aw\rangle - 2s\langle w, Ag\rangle + s^2\langle g, Ag\rangle}{\| v - sg \|_B^2} \\ &\leq \frac{|||w|||^2 - 2s\eta_\tau |||w||| \cdot |||g||| + s^2 |||g|||^2}{(\| v \|_B - s \| g \|_B)^2} \\ &\leq \frac{1-\eta_\tau^2}{(1-\eta_\tau\beta)^2}\mathrm{RQ}(v), \end{aligned}$$

where $\beta = (\| g \|_B \cdot |||w|||)/(|||g||| \cdot \| v \|_B)$. ¿From (6) and (8), it follows that

$$\begin{aligned} \beta &\leq \frac{\theta \| g_\perp \|_B}{|||g_\perp|||}\sqrt{\mathrm{RQ}(v)} \\ &\leq \theta_\tau. \end{aligned}$$

Hence,

$$\mathrm{RQ}(v - sq) \leq \gamma_\tau \mathrm{RQ}(v).$$

It has thus been shown that choosing a certain correction, $-sg$, in $\mathcal{P}$ reduces the Rayleigh quotient at least by a factor of γ_τ. Since Step 2 chooses the optimal correction in $\mathcal{G}$, (9) must be true and the theorem is proved. □

REMARK. *The premise of this theorem that* τ *is sufficiently small is used to assure that* $(v_1, u) \neq 0$ *and that* γ_τ *makes sense. The latter assertion is certainly true when* $\tau \leq \min\{\kappa, 1/(\sqrt{\eta}\theta)\}$*, where* $\kappa = \sqrt{\eta}/\zeta(1+\sqrt{\eta}))$*. The former is also true for such* τ *because* $\tau \leq \kappa < 1$*, which means that* $R(v_1) - \lambda < \gamma$ *and, hence* $(v_1, u) \neq 0$.

The next theorem provides general estimates for the inequalities (5)-(7). For this purpose, suppose that the original composite grid operator is given by $A_0 = A + \lambda_1 B$, where $\lambda_1 > 0$. Note that (8) holds for $\gamma = \lambda_2 - \lambda_1 > 0$, where λ_2 is the second eigenvalue of (A_0, B). Let $\rho(\cdot)$ denote spectral radius and *id* the identity operator (on the appropriate space), and define the following quantities:

$A_0^I = I^T A_0 I$ [coarse-grid Galerkin approximation to A_0];

$A^I = I^T A I$ [coarse-grid Galerkin approximation to A; A^I is nonsingular because $u \notin \mathcal{G}$];

$B^I = I^T B I$ [coarse-grid Galerkin approximation to B];

$|||v|||_0 = \langle v, A_0 v\rangle^{1/2}$;

$u_0 \in \mathcal{H} = A_0$=unit eigenvector of (A_0, B) for the eigenvalue $\lambda_1 [u = \lambda_1 u_0]$;

$u_0^I \in G = A_0^I$-unit eigenvector of (A_0^I, B^I) for its smallest eigenvalue λ_1^I [which for convenience is assumed to be simple];

$G_\perp = \{w \in G : \langle w, B^I u_0^I\rangle = 0\}$ [coarse-grid analog of $\mathcal{H}_\perp$];
$I_\perp = (id - u_0 u_0^T A_0) I (id - u_0^I (u_0^I)^T A_0^I)$ [A-orthogonal projection onto $\mathcal{H}_\perp$ of I restricted to $G_\perp$];
$Q^\perp = id - I_\perp (A^I)^{-1} I_\perp^T A$ [A-orthogonal projection of $\mathcal{H}^\perp$ onto the A-orthogonal complement of $I_\perp G_\perp$;
$Q_0^\perp = id - I(A_0^I)^{-1} I^T A_0$ [A_0-orthogonal projection of $\mathcal{H}$ onto the A-orthogonal complement of $I_\perp G_\perp$];
$\delta = \rho(Q^\perp A^{-1})\rho(A^I)[Q^\perp A^{-1}$ is restricted to $A\mathcal{H}^\perp$];
$\delta_0 = \rho(Q_0^\perp A_1^{-1}\rho(A_0^I)$;
$\sigma = \mathrm{cond}(B^{1/2}) = \| B^{1/2} \| \cdot \| B^{-1/2} \|$;
$\kappa = |||u_0 - I u_0^I|||_0^2$;
$\nu = \| q_0 \|_B \,/\, \| u_0 \|_B$ (where q_0 is the B-orthogonal projection of u_0 onto $\mathcal{P}$).

The quantity δ_0 appears in many modern theoretical estimates of convergence of multigrid methods applied to linear equations, starting with [Maitre and Musy 1984]. See also [Mandel, McCormick, and Bank 1987]. It is used to prove convergence of FAC for linear equations in [McCormick 1984]. The first task in obtaining estimates for the assumptions in (5)-(7) is to relate δ to δ_0, which is done in the following lemma. This will then be used in Theorem 2 as a basis for establishing (5). Theorem 2 will also use this lemma, together with the quantity κ and the definition of δ that crucially involves the space $IG_\perp$, to establish (6). Theorem 2 will establish (7) using the quantity ν. Finally, for a class of problems, Theorem 3 in Section 5 will prove mesh independent convergence of RQMG by establishing bounds on the necessary quantities (e.g., δ_0, σ, and κ.)

LEMMA. *Let*

$$\omega = \left(1 + \frac{\lambda_1 \sigma}{\gamma}\right)^2 \left(1 + \frac{k \| B^{-1} \|}{\rho(Q_0^T A_0^{-1})\lambda_1}\right).$$

Then

$$\delta \le \omega \delta_0. \tag{14}$$

PROOF: The definition of A_0 and the nonnegativity of A show that

$$\begin{aligned} \rho(A^I) &= \rho(A_0^I - \lambda_1 B^I) \\ &\le \rho(A_0^I). \end{aligned}$$

It therefore suffices to prove that

$$\rho(Q^\perp A^{-1}) \le \omega \rho(Q_0^T A_0^{-1}). \tag{15}$$

To this end, note that since $Q^\perp$ is an A-orthogonal projector, then $Q^\perp A^{-1}$ is symmetric and

$$\rho(Q^\perp A^{-1}) \quad = \sup_{0 \ne r \in A\mathcal{H}_\perp} \frac{\langle Q^\perp A^{-1} r, r\rangle}{\langle r, r\rangle}$$

$$= \sup_{0 \neq v \in \mathcal{H}_\perp} \frac{\langle Q^\perp v, Av \rangle}{\langle Av, Av \rangle}$$

$$= \sup_{0 \neq v \in \mathcal{H}_\perp} \frac{|||Q^\perp v|||^2}{\| Av \|^2}.$$

Hence, by the definition of $Q^\perp$,

$$\rho(Q^\perp A^{-1}) = \sup_{0 \neq v \in \mathcal{H}_\perp} \inf_{g \in IG_\perp} \frac{|||v - g|||^2}{\| Av \|^2}. \tag{16}$$

Similarly,

$$\rho(Q_0^\perp A_0^{-1}) = \sup_{0 \neq v \in \mathcal{H}} \inf_{g \in IG} \frac{|||v - g|||_0^2}{\| A_0 v \|^2}. \tag{17}$$

Let $0 \neq v \in \mathcal{H}_\perp$ be arbitrary and denote by g_0 the element of $\mathcal{G}$ guaranteed by (17) to satisfy

$$\frac{|||v - g_0|||_0^2}{\| A_0 v \|^2} \leq \rho(Q_0^\perp A_0^{-1}). \tag{18}$$

Now write

$$g_0 = g + \beta I u_0^I, \tag{19}$$

where $g \in IG_\perp$. Then, for some A_0-unit vector in $w \in \mathcal{H}_\perp$, it follows that

$$I u_0^I = a u_0 + b w, \tag{20}$$

where, by definition of κ,

$$b^2 \leq \kappa, \qquad a^2 \geq 1 - \kappa. \tag{21}$$

Note that $A_0 \geq \lambda_1 B$ (by which is meant that $\langle z, A_0 z \rangle \geq \lambda \langle z, Bz \rangle$ for all z), from which follows

$$\langle v, A_0 v \rangle \leq \frac{1}{\lambda_1} \langle (A_0^{1/2} B^{-1} A_0^{1/2}) A_0^{1/2} v, A_0^{1/2} v \rangle.$$

Hence,

$$|||v|||_0^2 \leq \frac{\| B^{-1} \|}{\lambda_1} \| A_0 v \|^2 . \tag{22}$$

Thus, (18), (19), (20), (22), the Cauchy-Schwarz inequality, and the fact that the A_0-unit vector w is in $\mathcal{H}_\perp$ imply that

$$\begin{aligned}
\| A_0 v \|^2 \, \rho(Q_0^\perp A_0^{-1}) &\geq |||v - g_0|||_0^2 \\
&= |||v - g|||_0^2 - 2\beta b \langle v, A_0 w \rangle + \beta^2 |||I u_0^I|||_0^2 \\
&\geq |||v - g|||_0^2 - b^2 \langle v, A_0 w \rangle^2 \\
&\geq |||v - g|||_0^2 - \kappa |||v|||_0^2 |||w|||_0^2 \\
&\geq |||v - g|||_0^2 - \frac{\kappa \| B^{-1} \|}{\lambda_1} \| A_0 v \|^2 .
\end{aligned}$$

(β is eliminated in the third line by noting that $\beta^* = b\langle v, A_0 w\rangle$ minimizes the expression in the second line over β. Remember that $|||Iu_0^I|||_0 = 1$.) Thus,

$$\frac{|||v-g|||_0^2}{\| A_0 v \|^2} \leq \rho(Q_0^\perp A_0^{-1}) + \frac{\kappa \| B^{-1} \|}{\lambda_1}. \tag{23}$$

But $A_0 = A + \lambda_1 B \geq A$, so

$$|||v-g||| \leq |||v-g|||_0. \tag{24}$$

Since $v \in \mathcal{H}_\perp$, then

$$\begin{aligned} \| A_0 v \| &\leq \| Av \| + \lambda_1 \| Bv \| \\ &\leq \| Av \| + \lambda_1 \| B^{1/2} \| \cdot \| B^{1/2} v \| . \end{aligned} \tag{25}$$

But $B^{1/2}v$ is in the space spanned by the conventional eigenvectors of $B^{-1/2}AB^{-1/2}$ belonging to positive eignvalues, the smallest of which is γ. Hence,

$$\begin{aligned} \| B^{1/2} v \| &\leq \frac{1}{\gamma} \| (B^{-1/2}AB^{-1/2})B^{1/2}v \|^2 \\ &\leq \frac{1}{\gamma} \| B^{-1/2} \| \| Av \| . \end{aligned} \tag{26}$$

Thus, (25) and (26) combine to yield

$$\| Av \| \geq \left(1 + \frac{\lambda_1 \sigma}{\gamma}\right)^{-1} \| A_0 v \| . \tag{27}$$

(15) now follows from (16), (23), (24), and (27), so the lemma is proved. □

Theorem 2. *(5) holds for* $\eta = \dfrac{1}{1 + t\omega\delta_0}$, *where* $t = \dfrac{1}{1 - (\lambda^I \kappa)^{1/2}\sigma}$; *(6) holds for* $\theta = \left(1 - \left(\dfrac{\lambda_1 \kappa}{\lambda_2(1-\kappa)}\right)^{1/2}\right)^{-1}$; *and (7) holds for* $\zeta = (1-v^2)^{-1/2}$.

PROOF: To prove (5) for this η, let $y \in \mathcal{Y}$ and note that

$$\| B^I \| = \| I^T B I \| \leq \| B \| \| I \|^2 = \| B \| .$$

Hence, by (20)-(21) and the fact that u_0^I is an A_0^I-unit vector, we have

$$\begin{aligned} \| A_0^I u_0^I (Iu_0^I)^T Ay \| &= \| A_0^I u_0^I \| \, |(Iu_0^I)^T Ay| \\ &= \| B^I \|^{1/2} \| (B^I)^{-1/2} A_0^I u_0^I \| \, |\langle bw, Ay\rangle| \\ &\leq (\lambda_1^I \| B^I \| \kappa)^{1/2} \| w \| \| Ay \| \\ &\leq (\lambda_1^I \kappa)^{1/2} \sigma \| B^{1/2} w \| \| Ay \| \\ &= (\lambda_1^I \kappa)^{1/2} \sigma \| I^T Ay \|, \end{aligned}$$

where the last line follows from the fact that Ay is zero on the patch. Hence, using the identity $I^T A = ((id - u_0 u_0^T A_0)I)^T A$, we have

$$\begin{aligned}
\| I^T Ay \| &\leq \frac{1}{1-(\lambda_1^I \kappa)^{1/2}\sigma}(\| I^T Ay \| - \| A_0^I u_0^I (u_0^I)^T I^T Ay \|) \\
&\leq t \| (id - A_0^I u_0^I (u_0^I)^T) I^T Ay \| \\
&= t \| I_\perp^T Ay \| .
\end{aligned}$$

Now define $Q = I_\perp (A^I)^{-1} I_\perp^T A$, the A-orthogonal projection of $\mathcal{H}_\perp$ onto $I_\perp G_\perp$. Then, following the proof in [McCormick 1984], and using the above inequality, we have

$$\begin{aligned}
|||y|||^2 - |||Qy|||^2 &= |||Q^\perp y|||^2 \\
&= \langle Q^\perp y, AQ^\perp y\rangle \\
&= \langle Q^\perp y, Ay\rangle \\
&\leq \rho(Q^\perp A^{-1}) \| Ay \|^2 \\
&= \rho(Q^\perp A^{-1}) \| I^T Ay \|^2 \\
&\leq t\rho(Q^\perp A^{-1}) \| I_\perp^T Ay \| \\
&\leq t\rho(Q^\perp A^{-1})\rho(A^I)\langle (A^I)^{-1} I_\perp^T Ay, I_\perp^T Ay\rangle \\
&= t\delta\langle I_\perp (A^I)^{-1} I_\perp^T Ay, Ay\rangle \\
&= t\delta\langle Qy, Ay\rangle \\
&= t\delta |||Qy|||^2 \\
&\leq t\omega\delta_0 |||Qy|||^2,
\end{aligned}$$

where the last inequality uses (14). Hence,

$$|||Qy|||^2 \geq \frac{1}{1+t\omega\delta_0}|||y|||^2.$$

But then

$$\begin{aligned}
|||Q^\perp y|||^2 &= |||y|||^2 - |||Qy|||^2 \\
&\leq (1-\eta)|||y|||^2.
\end{aligned}$$

This implies (5), as asserted.

To prove (6), let g be the element of $IG_\perp$ satisfying (5). Then (20),(21), and the relations $(u_0, u_0) = \lambda_1^{-1/2} \| u_0 \|_B$ and $\| w \|_B \leq \lambda_2^{-1/2}$ yield

$$\begin{aligned}
\| g_\perp \|_B &= \| g - \frac{(g, u_0)}{(u_0, u_0)} u_0 \|_B \\
&= \| g - \frac{\lambda_1^{1/2}(g, Iu_0^I - bw)}{a \| u_0 \|_B} u \|_B
\end{aligned}$$

$$
\begin{aligned}
&= \| g - \frac{b\lambda_1^{1/2}(g,w)}{a \| u_0 \|_B} u_0 \|_B \\
&\geq \| g \|_B - \left| \frac{b\lambda_1^{1/2}}{a} \right| \| g \|_B \| w \|_B \\
&\geq \left(1 - \left(\frac{\lambda_1 \kappa}{\lambda_2 (1-\kappa)} \right)^{1/2} \right) \| g \|_B \\
&= \theta^{-1} \| g \|_B \ .
\end{aligned}
$$

This proves (6).

To prove (7), note that the definition of v implies that

$$
\begin{aligned}
\sup_{p \in \mathcal{P}} \frac{(p, u_0)2}{\| p \|_B^2 \| u_0 \|_B^2} &= \frac{(q_0, u_0)^2}{\| q_0 \|_B^2 \| u_0 \|_B^2} \\
&= \frac{||q_0||_B^2}{||u_0||_B^2} \\
&= v^2.
\end{aligned}
$$

Thus, for any $p \in \mathcal{P}$, it follows that

$$
\begin{aligned}
\frac{\| p_\perp \|_B}{\| p \|_B} &= \frac{\| p - \frac{(p,u_0)}{(u_0,u_0)} u_0 \|_B}{\| p \|_B} \\
&= \left(1 - \frac{(p, u_0)^2}{\| p \|_B^2 \| u_0 \|_B^2} \right)^{1/2} \\
&\geq \frac{1}{\zeta}
\end{aligned}
$$

This proves (7) and, hence, the theorem. □

5. Applications

This section will apply the theory of Section 4 to the conventional eigenproblem for a general class of second-order self-adjoint positive definite operators. Let L be such an operator applied to a Hilbert space of functions defined on a suitable multidimensional domain, in such a way that L has full regularity. Assume that the smallest eigenvalue of L is simple and that the support of its eigenvector is not contained wholly within the refinement region. The conventional eigenproblem for L is discretized (perhaps in the weak or variational sense) by a Rayleigh-Ritz method applied to the Rayleigh quotient, using a linear or multilinear finite element space. For simplicity, the elements are assumed to be piecewise linear functions corresponding to a composite grid that consists of a uniform global grid of mesh size h and a local, uniform, conforming refinement using a fixed mesh refinement factor. Let A_0 be the resulting stiffness matrix, B the resulting mass matrix, and λ_1 and λ_2 the respective smallest and second smallest eienvalues of (A_0, B). Then there exists a constant $c \ll \infty$ such that

$$\lambda_1 \parallel B \parallel \ll \infty, \tag{28}$$

$$\sigma \ll \infty, \tag{29}$$

$$\frac{\gamma}{\lambda_1} \ll \infty, \tag{30}$$

$$\rho(Q_0^T A_0^{-1}) \leq ch^2, \tag{31}$$

$$\rho(A_0^I) \leq ch^{-2}, \tag{32}$$

where by the expression $a \ll \infty$ we mean that a is bounded above uniformly in h. That (28)-(30) are true follows directly from assumptions on the differential eigenproblem and standard estimates for its discretization; (31) and (32) are estimates obtained in a standard way from finite element theory (cf. [Maitre and Musy 1984], [Greenbaum 1984], and [Mandel, McCormick, and Bank 1987]). These estimates can be used to establish the following.

THEOREM 3. *For small enough h, RQMG converges linearly to the composite grid approximation to the eigenvector of L belonging to its smallest eigenvalue. The linear factor is bounded above by a constant less than one uniformly in h.*

PROOF: Inspection of the constants ω, η, θ, and ζ and the bounds (5)-(7) and (14) show that the proof would follow from (28)-(32) and the following (for sufficiently small h and a possibly larger constant $c \ll \infty$):

$$\frac{\kappa \parallel B^{-1} \parallel}{\rho(Q_0^\perp A_0^{-1})\lambda_1} \ll \infty, \tag{33}$$

$$\kappa \leq ch^2, \tag{34}$$

$$\frac{1}{1 - v^2} \ll \infty. \tag{35}$$

To prove (33), let g_0 be the element of $\mathcal{G}$ that, by (17), is guaranteed to satisfy

$$\frac{|||u_0 - g_0|||_0^2}{\parallel A_0 u \parallel^2} \leq \rho(Q_0^\perp A_0^{-1}).$$

Then, letting $\varepsilon_1 = \lambda_1 \parallel B \parallel \rho(Q_0^T A_0^{-1})$ (note by (28) and (31) that ε_1 is arbitrarily small) yields

$$\begin{aligned} |||u_0 - g_0|||_0^2 &\leq \parallel \lambda_1 B u_0 \parallel^2 \rho(Q_0^\perp A_0^{-1}) \\ &\leq \lambda_1 \parallel B \parallel \langle \lambda_1 B u_0, u_0 \rangle \rho(Q_0^\perp A_0^{-1}) \\ &= \varepsilon_1. \end{aligned}$$

Now writing $g_0 = \alpha u_0 + \beta w$, where w is an A_0-unit vector in $\mathcal{H}_\perp$, it follows from the above that

$$\beta^2 \le \varepsilon_1,$$
$$\alpha^2 \ge (1 - \varepsilon_1^{1/2})^2.$$

Thus, letting $\varepsilon_2 = \varepsilon_1(1-\varepsilon_1^{1/2})^{-2}$ (which is also arbitrarily small) yields

$$\begin{aligned}
\mathrm{RQ}(g_0) &= \frac{\alpha^2+\beta^2}{\alpha^2 \parallel u_0 \parallel_B^2 + \beta^2 \parallel w \parallel_b^2} - \lambda_1 \\
&\le \frac{\alpha^2+\beta^2}{\alpha^2/\lambda_1} - \lambda_1 \\
&= \lambda_1 \frac{\beta^2}{\alpha^2} \\
&\le \lambda_1 \varepsilon_2.
\end{aligned}$$

Now let $Iu_0^I = au_0 + bw_0$, where $a^2 + b^2 = 1$ and w_0 is an A_0-unit vector in $\mathcal{H}_\perp$. Then, since Iu_0^I minimizes $\mathrm{RQ}(g)$ over $g \in IG$, it follows that

$$\begin{aligned}
\lambda_1 \varepsilon_2 &\ge \mathrm{RQ}(Iu_0^I) \\
&= \frac{1}{(a^2/\lambda_1) + b^2 \parallel w_0 \parallel_B^2} - \lambda_1 \\
&\ge \left(\frac{1}{a^2 + \frac{\lambda_1}{\lambda_2} b^2} - 1 \right) \lambda_1 \\
&= \lambda_1 \frac{1 - \frac{\lambda_1}{\lambda_2}}{1 - (1 - \frac{\lambda_1}{\lambda_2}) b^2} b^2.
\end{aligned}$$

Hence,

$$b^2 \le \frac{\varepsilon_2}{(1 - \frac{\lambda_1}{\lambda_2})(1+\varepsilon_2)},$$

from which follows

$$\begin{aligned}
\kappa &= |||u_0 - Iu_0^I|||_0^2 \\
&= (1-a)^2 + b^2 \\
&= 2(1 - \sqrt{1-b^2}) \\
&\le 2 \left(1 - \left(1 - \frac{\varepsilon_2}{(1-\frac{\lambda_1}{\lambda_2})(\lambda_1 + \varepsilon_2)} \right)^{1/2} \right) \\
&\le \frac{2\varepsilon_2}{(1-\frac{\lambda_1}{\lambda_2})(1+\varepsilon_2)}.
\end{aligned}$$

By (31) and the definitions of ε_1 and ε_2, for sufficiently small h it follows that there exists a constant $c_1 \ll \infty$ such that

$$\frac{\kappa \parallel B^{-1} \parallel}{\rho(Q_0^T A_0^{-1})\lambda_1} \leq c_1 \sigma_0.$$

With (29), this proves (33). (34) follows for sufficiently small h from noting that $\kappa \leq c_1 \lambda_1 \parallel B \parallel \rho(Q_0^T A_0^{-1})$ and using (28) and (31).

Now (35) follows from noting that: u_0 interpolated to the Hilbert space is an arbitrarily accurate approximation to the unit eigenvector of L belonging to its smallest eigenvalue in the energy norm, and, consequently, by a standard estimate in the Hilbert space norm; thus, $\parallel q_0 \parallel_B / \parallel u_0 \parallel_B$ must approach the corresponding ratio for the eigenvector of L, which is assumed to be less than one. This proves (35) and, hence, the theorem. □

REMARK. *This theory can be rather easily extended in several ways. For example, it applies with little change to higher order differential operators, provided the order of the element spaces is increased to ensure the usual approximation property. Also, by appropriate accounting of scale (i.e., defining the discrete inner product to approximate the Hilbert space inner product), this theory can be extended to the case that the refinement factor (i.e., ratio of coarse to fine mesh sizes) is arbitrarily large. Finally, the theory can be extended in a straightforward way to the case that the global and patch grids are nonuniform, similar to the extensions developed in [McCormick and Thomas 1986]. However, the special case treated here was chosen to simplify what would otherwise be somewhat cumbersome notation.*

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