ALGEBRAIC MULTIGRID (AMG) FOR GEODETIC SURVEY PROBLEMS

by

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1. Introduction

Algebraic multigrid (AMG) is a method which applies the basic multigrid ideas to discrete systems of equations. Multigrid methods are generally applied to problems arising from the discretization of partial differential equations. However, many other problems, for which application of standard multigrid techniques is difficult or impossible, share many properties with those discrete systems and can benefit from multigrid type processes. Decisions on the grids and grid transfer operators can be based on the operator entries and structure, and a sequence of coarser grids along with the interpolation and coarse grid operators can be constructed. Multigrid type cycling algorithms can then be applied to yield a fast solution to the problem.

The most critical point in determining a successful AMG algorithm is the choice of interpolation.

This is actually a number of choices: How many and which points to use for interpolation to a particular point, which are the coarse grid points, and what interpolation formula should be used. These are interconnected, but here the main concern is the third question. Once the first two are resolved, the goal is to pick some formula which results in the range of interpolation closely approximating the smooth functions. Here, we define smooth to mean the type of error which results after relaxation. In many geometrically based problems, particularly those of positive type, this coincides with the usual MG definition of smooth. That is, it is well approximated by a linearly interpolated coarse grid function. The goal of AMG however is to determine the processes based on information contained in the given matrix operator. This has been successfully accomplished for positive type operators. However for those of non-positive type, specifically those arising from geodetic survey problems with an unknown reference direction, the idea of smoothness must be refined. Several approaches have been tried. A straightforward application of the operator interpolation described in section 2 fails. An examination of the interpolation shows the reason for this. Four methods to resolve this problem are given. The first relies almost entirely on geometrical information. The second also uses geometrical information, but is a refinement of operator interpolation. The third is independent of the geometry, but requires a different formulation of the problem. The last is a general approach to the idea of smoothness which uses only the operator. Each has some applications, depending on what is known about the problem in question.

2. AMG for positive type systems

The basic ideas of AMG are briefly described here. Let A be an $n \times n$ symmetric matrix of positive type. That is, $A = \{a_{ij}\}$ with $a_{ii} > 0$, $a_{ij} < 0$ for $j \neq i$, and $\sum_j a_{ij} = 0$ for $i = 1, 2, \cdots, n$. In addition, let A be "local" in some sense. Now consider the problem $Au = f$. Relaxation on such a system will reduce the residual and smooth the error. (see [1]) Then the equations hold approximately and, for a point i, the error e between the exact solution and the current approximation u satisfies the equation

$$\sum_j a_{ij} e_j \approx 0.$$

If we know the error at all points $j \neq i$ for which $a_{ij} \neq 0$, then we know e_i must satisfy

$$e_i = -\frac{1}{a_{ii}} \sum_{j \neq i} a_{ij} e_j .$$

That is, e_i is a weighted average of e at surrounding points. In fact, we can generally express e_i as a weighted average of a smaller number of points. The number will depend on several factors, such as the dimension of the underlying problem, if it does possesses one, and the relative size of the matrix entries. Denoting this set of points by $S(i)$, we can write

(1) $e_i = -\left(a_{ii} + \sum_{\substack{j \notin S(i) \\ j \neq i}} a_{ij}\right)^{-1} \sum_{k \in S(i)} a_{ik} e_k .$

(The denominator is not written as $\sum_{j \notin S(i)} a_{ij}$, even though they are equivalent in this case, because if the assumption $\sum_j a_{ij} = 0$ is relaxed, the resulting formula would not be correct.) The above formula is based on the smoothness assumption

(2) $\sum_{j \notin S(i)} a_{ij} (e_i - e_j) = 0 .$

If the points $1, 2, \cdots, n$ are partitioned into two sets $\mathcal{C}$ and $\mathcal{F}$, known as coarse and fine points respectively, so that $i \in \mathcal{F}$ implies that $S(i) \subset \mathcal{C}$, then the error at all points in $\mathcal{F}$ can be found if errors at points in $\mathcal{C}$ are known. Let $\mathcal{Y} = \{1, 2, \cdots, n\}$, and e^c be a function defined on points in $\mathcal{C}$. Let $F(\mathcal{G})$ and $F(\mathcal{C})$ be the set of all functions defined over $\mathcal{G}$ and $\mathcal{C}$ respectively, and let subscripts denote point evaluation. Then the interpolation operator $I_c^f : F(\mathcal{C}) \to F(\mathcal{G})$ can be defined so that

$$(I_c^f e^c)_i = \begin{cases} e_i^c & \text{if } i \in \mathcal{C}, \text{ and} \\ -\left(a_{ii} + \sum_{\substack{j \notin S(i) \\ j \neq i}} a_{ij}\right)^{-1} \sum_{k \in S(i)} a_{ik} e_k^c & \text{if } i \in \mathcal{F} \end{cases}$$

The process of applying this operator will be called operator interpolation, since it is based on the matrix.

Once relaxation has been performed, the error will lie approximately in the range of interpolation, and the error in the range can be eliminated by solving the equation

$$(3) \quad (I_c^f)^T A I_c^f e^c = (I_c^f)^T (f - Au)$$

and updating the current approximation u by

$$u \leftarrow u + I_c^f e^c.$$

The coarse grid equation is obtained by finding that e^c which minimizes the functional

$$\langle A(u + I_c^f e^c) - 2f,\ u + I_f^c e^c \rangle.$$

The operator on the left side of (3) is the coarse grid operator

$$A^c = (I_c^f)^T A I_c^f.$$

This process is applied recursively, obtaining a sequence of operators $A = A^1, A^2, \cdots, A^M$. The problem on level k is solved by relaxation and appealing to level k+1 in order to find the error on level k. If $k = M$, the problem is solved exactly. M is taken large enough so that A^M is small enough to make the solution process cheap.

3. The Geodetic Survey Problem

An example of a problem which yields a matrix equation not of positive type is the geodetic survey problem. Here, a simplified version of that problem is explained. Further simplifications will be introduced later in this section in order to make the principles involved clearer.

Suppose n points are located in the plane with positions (X_i, Y_i), $i = 1, 2, \cdots, n$. Approximations (x_i°, y_i°) to these positions are given. Each point has a reference direction which is approximately in the positive y direction. The angle between the y-axis and this reference direction for a point i is denoted by z_i. Then, for each point i, readings are made measuring the angle between the reference direction and the line passing through point i and point j for all j in some set of neighbors $N(i)$ of point i. For convenience, we require that $j \in N(i)$ implies that $i \in N(j)$. The problem is to find the vectors $u_i = (x_i, y_i, z_i)^T$, $i = 1, 2, \cdots, n$, such that the sum of the squares of the angle measurement errors is minimized. Each of the error terms is a non-linear function of x and y. This can be linearized around the given approximations to the positions. The problem is then to minimize a functional of the form

$$F(x,y,z) = \sum_i \sum_{j\in N(i)} \left(\alpha_{ij}(x_i - x_j) + \beta_{ij}(y_i - y_j) + z_i + \gamma_{ij}\right)^2.$$

Now, to simplify the problem further, let y_i be fixed at i, $i = 1, 2, \cdots, n$, and let $N(i)$ be the set $\{j : j \neq i,\ \max\{1, i-2\} \leq j \leq \min\{n, i+2\}\}$. In addition, force $|x_i|, |x_i^0| \ll 1$ for all i. This gives a new functional to be minimized,

$$F(x,z) = \sum_i \sum_{j\in N(i)} \left(\alpha_{ij}(x_i - x_j) + z_i + \gamma_{ij}\right)^2.$$

Now, letting $u = (u_1, u_2, \cdots, u_n)^T$ with $u_i = (x_i, z_i)^T$, the function u which minimizes F is the solution to a problem $Au = f$, where A is an $n\times n$ block matrix where the blocks are 2×2, and $f = (f_1, \cdots, f_n)^T$ with $f_i = (f_i^x, f_i^z)^T$. The equation for u_i, $3 \leq i \leq n-2$, is approximately:

$$(4)\quad \begin{bmatrix} \tfrac12 & \tfrac12 \\ -\tfrac12 & 0 \end{bmatrix}\begin{bmatrix} x_{i-2} \\ z_{i-2} \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x_{i-1} \\ z_{i-1} \end{bmatrix} + \begin{bmatrix} -5 & 0 \\ 0 & -4 \end{bmatrix}\begin{bmatrix} x_i \\ z_i \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_{i+1} \\ z_{i+1} \end{bmatrix} + \begin{bmatrix} \tfrac12 & -\tfrac12 \\ \tfrac12 & 0 \end{bmatrix}\begin{bmatrix} x_{i+2} \\ z_{i+2} \end{bmatrix} = \begin{bmatrix} f_i^x \\ f_i^z \end{bmatrix}.$$

4. AMG and the Geodetic Survey Problem

4.1 Grids and Relaxation. In the simplified geodetic problem of the last section, let $n = 2^M + 1$. Two point interpolation is reasonable here, so a natural coarsening method is to pick alternating points as C points. G^1, the fine grid, is $\{1, 2, \cdots, n\}$. The coarse grids can then be given as follows for $k = 1, 2, \cdots, M-1$: let $C^k = \{ j \in G^1 : j = 2^k l + 1$ for some $l \}$, $G^{k+1} = C^k$, and $F^k = G^k - C^k$. The relaxation process used is block Gauss-Seidel, where the 2x2 system corresponding to the ith point is solved with i ranging over G^k.

4.2 Operator Interpolation. In order to define the coarse grid operators, an interpolation scheme must be chosen. The goal here, as mentioned before, is to choose a scheme so that the range of interpolation is approximately the range of relaxation. The block Gauss-Seidel relaxation used produces error which is smooth in the usual sense. Consider operator interpolation on this smooth error. Here, let $u_i = (x_i, z_i)^T$ denote the error at point i rather than the current approximation. If i is an $\mathcal{F}$ point, then $S(i) = \{i+1, i-1\}$, and application of operator interpolation (1) to (4) yields

$$\begin{bmatrix} x_i \\ z_i \end{bmatrix} = \begin{bmatrix} 1/2 & 1/4 \\ -1/4 & 0 \end{bmatrix} \begin{bmatrix} x_{i-1} \\ z_{i-1} \end{bmatrix} + \begin{bmatrix} 1/2 & -1/4 \\ \frac{1}{4} & 0 \end{bmatrix} \begin{bmatrix} x_{i+1} \\ z_{i+1} \end{bmatrix}.$$

Compare the equation for z_i above with the z_i equation in (4). They are, respectively,

(5a) $z_i = 1/4\,(x_{i+1} - x_{i-1})$ and

(5b) $z_i = 1/4\,(x_{i+1} - x_{i-1}) + 1/8\,(x_{i+2} - x_{i-2})$.

Note that both terms in the second equation are approximations to half the derivative of x at i. Thus the first equation, lacking the second term, gives a z_i which is only half what it should be. The error lies in the smoothness assumption (2) which in this case says that

$$1/2\,(x_{i+2} - x_i) + 1/2\,(x_i - x_{i-2}) = 0.$$

Computational results also show that operator interpolation fails here. The convergence rate per cycle approaches 1. In this case, the smoothness assumption must be revised.

4.3 <u>Geometric Interpretation</u>. The first interpolation scheme which seeks to remedy the problem with the smoothness assumption can be viewed as a purely geometric approach. The observation that in (5b) z_i is an approximation to the derivative of x at i is the basis for this method. The interpolation equation for z_i on level k is then given by

$$z_i = -2^{-k}(x_{i-2^{k-1}} - x_{i+2^{k-1}})$$

where $i \in \mathcal{F}^k$, and $i-2^{k-1}$ and $i+2^{k+1} \in \mathcal{C}^k$ are the adjoining $\mathcal{C}$ points.

The presence of a derivative term allows for a higher order interpolation in x, given by

$$x_i = \tfrac{1}{2}(x_{i+2^{k-1}} + x_{i-2^{k-1}}) - 2^{k-3}(z_{i+2^{k-1}} - z_{i-2^{k-1}}).$$

This approach worked quite well for the problem described. However, if $\max_i \{|X_i|\}$ was allowed to increase, and X_i changed much from point to point, convergence slowed considerably. This is because z was no longer a good approximation to the derivative of x.

4.4. Geometry and the Smoothness Assumption. If the smoothness assumption is modified using the geometrical interpretation of smooth, then a modification of operator interpolation is obtained. Note that (1) is obtained when the assumption $u_j = u_i$ for $j \notin S(i)$ is used for substitution into the operator equation for u_i. Here, we give the new smoothness assumption in a similar form:

$$z_{i+2} = z_{i-2} = z_i \quad \text{and}$$

$$x_{i+2} = 2x_{i+1} - x_i \quad , \quad x_{i-2} = 2x_{i-1} - x_i .$$

That is, x_{i+2} is extrapolated from x_{i+1} and x_i. This is easily done on all levels. The new interpolation formula coincides with the previous one on the fine grid away from the boundary. This method, with modifications in the extrapolation formula, works well for more general problems such as those with widely varying X_i's and unevenly spaced Y_i's. Approximations to the positions of the points are needed here, which is not a big problem with the geodetic problem, but can be a drawback on other applications where such information is not available.

4.5 <u>Functional Interpolation</u>. Another approach is possible with a least squares problem when the original functional is available. Given $i \in \mathcal{F}$, a new functional consisting only of those terms which contain points in $S(i)$ can be constructed. Assuming that the original readings were fairly consistent, after relaxation the current approximation at point i approximately minimizes this new functional, as well as the original one. Thus the equation for point i in the matrix problem generated by this functional can be used to define interpolation in terms of points in S(i). Only i and points in S(i) appear in this equation, so no smoothness assumption is necessary.

This can't be done on coarser grids since a functional with terms consisting of pairwise observations is not available. However, in this case such observations can be constructed due to the nature of the problem. There are basically two ways to create such observations. Since the grid 2 operator is already defined in terms of functional interpolation, coarse grid observations can be sought which reflect the fine grid observations in weight and form, and which, if used to construct an operator, would result in one that approximates the given coarse matrix. Another, which is generally less work, is to ignore the form and construct a set of observations $\{u_i^c = b_{ij} u_j^c\}$ These are constructed by taking fine grid observations and replacing F values with the C values from which the F point is interpolated, one at a time. These are written in the form $u_i = b''_{ij} u_j$, and if i and j appear more than once, an average is taken with weights obtained from the fine grid operator. The weight and form are reconstructed using the coarse grid operator itself.

The first method has not been explored, though it may offer some interesting possibilities. The generated observations could be used to obtain a coarse grid operator in the same way the fine grid operator was obtained from the original functional.

The second method has been tried. Once

the coarse grid observations have been constructed, they can be put into the form

$$c_{ij} u_i = a^c_{ij} u_j .$$

This is done by multiplying the observation $u_j = b_{ji} u_i$ by a^c_{ij}, so that $c_{ij} = a^c_{ij} b_{ji}$. These observations can again be used in two ways. Since a^c_{ij} gives the strength of dependence of u^c_i on u^c_j, the observations can be used directly to define interpolation on the coarse grid to point i from points in $S^c(i)$ by

$$(6)\quad u^c_i = \Big(\sum_{j \in S^c(i)} c_{ij}\Big)^{-1} \sum_{j \in S^c(i)} a^c_{ij} u^c_j .$$

The second way is based on the idea that smooth functions on the coarse grid will satisfy the observations. This becomes the smoothness assumption for a modification of operator interpolation and the resulting formula, obtained by substituting the observations for $j \notin S^c(i)$ into the equation for point i is

(7) $u_i = -\left(a^c_{ii} + \sum_{j \notin S(i)} c_{ij}\right)^{-1} \sum_{j \in S(i)} a^c_{ij} u_j$.

Both of these work well, and require no geometric information. However, as stated previously, the original functional from which the problem is derived is needed. Another problem which limits the application of this method to the full geodetic problem is that, given u_i, it must be possible to determine u_j from the observations connecting the two points. With y unknown, this is not possible, although some extension of this method may be applicable to such a problem.

4.6 Determination of Smoothness. Another method, as yet untried but which holds the most promise in range of application is actually the simplest in concept. The idea is to obtain a number of functions u^l, $l = 1, 2, \ldots p$ for some $p > 1$, by starting with p initial guesses and relaxing a number of times with each on the problem $Au^l = 0$. Since smooth functions are defined as the type of error resulting from relaxation, the u^ℓ's are smooth by definition. At a point i, consider the vector $(u_i, u_{j_1}, u_{j_2}, \cdots, u_{j_m})^T$ of values of a function u evaluated at points i and $j_1, j_2, \cdots, j_m \in N(i)$. The smooth functions at point i constitute a subspace of the space of all such vectors. Assuming that p is large enough so that $u^1, u^2, \cdots u^p$ evaluated at these points spans the subspace of smooth functions there, then all the information necessary concerning smooth functions is available. A minimal number of points in $N(i)$ can be chosen in order to determine u_i. This set of points becomes $S(i)$. Then interpolation can be defined in one of two ways. The direct method is simply to write $u_i = \sum_{j \in S(i)} b_{ij} u_j$, and use the smooth functions $u^1, u^2, \cdots, u^p$ to determine the coefficients. This explicitly gives the interpolation formula at point i. The second method is to use the smooth functions to find a smoothness assumption

$$\sum_{j \notin S(i)} a_{ij} u_j = b_{ii} u_i + \sum_{j \in S(i)} b_{ij} u_j .$$

This is used along with the operator equation for point i to obtain the interpolation formula

$$u_i = -(a_{ii} + b_{ii})^{-1} \sum_{j \in S(i)} (a_{ij} + b_{ij}) u_j .$$

There are obviously many questions to be answered here, but this approach of deciding smoothness by relaxation is the most general of those discussed, and may work on a large class of problems not of positive type.

5. Results.

Four of the methods described above were tested. The methods, referred to here as A, B, C, and D, are defined as follows:

A: The geometrically based interpolation of 4.3.

B: The method of 4.4.

C: The interpolation of 4.5 given by (6), and

D: The indirect interpolation of 4.5 given by (7).

The problem was the simplified geodetic problem described in section 3 with fixed y, $n = 257$, and the X_i's random numbers between 0 and X_{max}. The values of X_{max} chosen were 0, .1, .5, 1, and 2. The cycling scheme used was the F-cycle, which is roughly defined as follows: For $k = M-1$, the F-cycle is simply a V-cycle. For $k = M-2, M-3, \cdots, 1$, an F-cycle is defined recursively as a relaxation sweep, transfer of the problem to level $k+1$, solved by an F-cycle followed by a V-cycle but omitting the initial relaxation work in the V-cycle on level $k+1$, correction of the level k approximation, and, finally, another level k relaxation sweep. The asymptotic convergence rates per cycle of the methods are given in the following table.

		X_{max} 0.0	0.1	0.9	1.0	2.0
METHOD	A	.033	.034	.54	.66	.75*
	B	.033	.035	.054	.11	.21
	C	.033	.034	.28	.33	.43
	D	.034	.082	.28	.32	.42

* Rate still increasing after 10 cycles

Methods A, B, C, and D all do well for $X_{max} = 0.0$ and 0.1. This is because all yield essentially the same interpolation. However, as X_{max} increases, A degrades rapidly. C and D give about the same results, except for $X_{max} = .1$. This may be due to the smoothness assumption involved reacting quickly to the inconsistency of the equations. In any case, it is clear that method B is superior to the others. The rate for $X_{max} = 2$ is acceptable, while the other methods do not give good rates.

The method of 4.6 may be still better than B, since actual smooth functions are found, while B, using geometric information, makes assumptions about smoothness

For further details on the application of AMG to geodetic problems, see [2].

REFERENCES

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