MULTIGRID SOLUTION OF BIFURCATION PROBLEMS

H. D. Mittelmann

A class of nonlinear eigenvalue problems is solved by the multigrid version of an algorithm particularly suited for this class. The algorithm is shown to compare favorably with standard methods for continuation along the solution curve. Usually this continuation is done on the coarsest grid. The multigrid method may then be used to compute fine-grid solutions in regions of special interest. Numerical results are given for reaction-diffusion problems.

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ABSTRACT

A class of finite-dimensional nonlinear eigenvalue problems is considered which in general are derived as discretizations of nonlinear elliptic eigenvalue problems. For following the solution paths a continuation strategy is proposed for a generalized inverse iteration algorithm. A multi-grid method is then presented using continuation for the coarsest grid and refinement where desired in a nested iteration fashion. Numerical results show the continuation technique to be very robust and efficient. It compares favorably with a standard method for that purpose. Multigrid results are given for bifurcation from higher eigenvalues, for turning point problems on rectangular and on nonrectangular domains.

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1. Introduction

Nonlinear boundary value problems with several solutions and exhibiting bifurcation phenomena have only recently been treated with the numerical techniques which have been quite successful in the solution of boundary value problems; see, for example, [1,2,3,4,5,7,10,11,12,15]. Preconditioned conjugate gradient (PCCG) methods as well as multi-grid (MG) algorithms have been used extensively for linear problems and both have been applied to nonlinear problems, too, either after a linearization by, for example, Newton's method or in a suitable nonlinear version.

For the treatment of parameter-dependent nonlinear boundary value problems quite a few approaches have been proposed and used successfully; see, for example, [6,9,13]. Since there in general a sequence of linear or nonlinear problems has to be solved it is quite natural to combine these methods with the efficient PCCG and MG algorithms. Here we present such a combination. The underlying method is not classical, but it is the generalized inverse iteration of [10] for which a first combination with multi-grid ideas has been proposed in [11]. This method in its original form is applicable to a certain class of nonlinear eigenvalue problems. For this class, however, it proves to be a very robust and efficient method and it may also be generalized.

In the following we shall first describe the classes of problems treated, define the generalized inverse iteration and state a local convergence result. We propose then a simple but efficient way to use this algorithm for continuation purposes, i.e. for following the solution curves of a discrete nonlinear eigenvalue problem. Then a MG version of the method is given which is a generalization of the two-level method of [11,12]. Numerical results for the continuation strategy are compared to those obtained with the pseudo-arclength method as implemented in [1]. Finally, some experiments with the MG algorithm are reported.

The contents of the following sections are:

2. Nonlinear eigenvalue problems
3. The basic algorithm
4. The continuation method
5. A multi-grid algorithm
6. Extensions
7. Continuation results
8. Multi-grid results

2. Nonlinear Eigenvalue Problems

Instead of describing in detail the classes of continuous nonlinear eigenvalue problems which may be considered, we assume that by a suitable discretization with parameter h a finite-dimensional problem of the form

$$f(x) := \lambda Bx, \quad x \in R^n, \quad \lambda \in R, \quad n = n(h) \tag{1.1}$$

has been obtained, where B is a symmetric and positive definite n x n matrix and $f : R^n \to R^n$ a smooth mapping. This gives some preference to a discretization by finite element methods since these usually yield a symmetric and positive definite B as discretization of an elliptic operator. The assumption on B may, however, be relaxed (cf.6.).

Problem (1.1) is more general than that considered in [10] since there it was assumed that f is the gradient of a functional or equivalently that the matrix $F = f'$ is symmetric. Then solutions of (1.1) may be characterized as critical points of that functional and this was exploited in [10] to develop a global convergence theory for the generalized inverse iteration and in [12] to distinguish between relevant and irrelevant solutions of the discrete eigenvalue problem.

The symmetry of F together with that of B was also advantageous numerically in [10] since it allowed to use conjugate gradient methods particularly suited for the resulting linear systems. These methods were applied to the augmented system (cf. (2.1), (2.2)). That this technique in general is preferable over the combination of block-elimination [6] and conjugate gradient methods has also been confirmed for nonsymmetric systems by the numerical results in [3].

3. The basic Algorithm

The generalized inverse iteration for the solution of (1.1) iteratively computes from a given pair (x_k,λ_k), $\|x_k\|_B = \rho$,

$$x_{k+1} = \rho\tilde{x}_{k+1}/\|\tilde{x}_{k+1}\|_B,$$

$$\tilde{x}_{k+1} = x_k - H_k f(x_k), \tag{2.1a}$$

$$\lambda_{k+1} = f(x_{k+1})^T x_{k+1}/\rho^2$$

where

$$H_k = \begin{bmatrix} F_k - \lambda_k B & -Bx_k \\ -x_k^T B & 0 \end{bmatrix}^{-1}_{nxn}. \tag{2.1b}$$

Here $\|\cdot\|_B$ denotes the norm introduced by B. H_k is the nxn principal submatrix of the inverse of the matrix in brackets provided this is regular. We note a relationship to Newton's method for the augmented system

$$f(x) - \lambda Bx = 0, \tag{2.2}$$

$$-1/2\|x\|_B^2 + \rho^2/2 = 0.$$

The solutions of (1.1) are parametrized by their B-norm and (2.1) in contrary to Newton's method for (2.2) actually

generates only iterates with B-norm equal to ρ. The second difference is that the parameter λ is updated by a Rayleigh-quotient in (2.1). The starting values for (2.1) are thus assumed to be (x_1,λ_1) with $\|x_1\|_B = \rho$ and $\lambda_1 = f(x_1)^T x_1/\rho^2$.

We must expect in general the existence of multiple solutions with respect to λ and ρ. A globally convergent method as, for example, damped Newton's method for (2.2) or the global version of the generalized inverse iteration of [10] may converge to any of these solutions for the given parameter value. So, in order to stay on a solution branch we rather try to exploit the local convergence properties of an algorithm. The following result was proved in [10,12].

Theorem 2.1 Let f in (1.1) be twice continuously differentiable in a neighbourhood of a solution (x_0,λ_0) of (1.1) and assume that

$$N(F_0-\lambda_0 B) \subset \text{span}\{Bx_0\}.$$

Here N(L) denotes the nullspace of the linear operator L. Then for x_1, $\|x_1\|_B = \|x_0\|_B = \rho$ sufficiently close to x_0 the sequence $\{x_k\}$ generated by (2.1) converges quadratically to x_0.

4. The Continuation Method

The parametrization by ρ in the generalized inverse iteration is, of course, not as general as one by, for example, the pseudo-arclength along the solution curve. In the following we shall see, however, that many problems of the

form (1.1) may be solved very efficiently and reliably with a very simple ρ-continuation technique. This technique may be further refined and combined, for example, with methods to switch branches at bifurcation points or to compute singular points. In the following we describe the basic step of the ρ-continuation process.

First we derive some useful formulae.

Lemma 3.1 If the following expressions are well-defined it holds for solutions of problem (1.1) that

$$\frac{d\rho}{d\lambda} = x^T B \frac{dx}{d\lambda} \,/\rho \tag{3.1}$$

$$\frac{d(x/\|x\|_B)}{d\rho} = \left(\frac{dx}{d\lambda}\frac{d\lambda}{d\rho} - \frac{x}{\rho}\right)/\rho. \tag{3.2}$$

Proof We have $\rho^2 = f(x)^T x/\lambda$, so

$$\frac{d\rho}{d\lambda} = \left[\left(x^T F(x)\frac{dx}{d\lambda} + f(x)^T \frac{dx}{d\lambda}\right)\lambda - f(x)^T x\right]/(2\rho\lambda^2).$$

From (1.1) we have, except in turning points,

$$(F(x)-\lambda B)\frac{dx}{d\lambda} = Bx \tag{3.3}$$

from which (3.1) follows. But by differentiation and using (3.1) we derive (3.2).

If a solution (x_ρ, λ_ρ) for a given ρ-level is known and one for a different level $\rho + \delta\rho$ is to be computed then algorithm (2.1) requires the starting guess to have that norm. A very simple predictor-step is thus to use $x_\rho(\rho+\delta\rho)/\rho$ and the corresponding λ. For the quality of this guess the derivative (3.2) is important. We therefore propose to choose $|\delta\rho|$ inverse proportional to this derivative. With a suitable monotone function g let

$$|\delta\rho| = g\left(\left\|\frac{d(x/\|x\|_B)}{d\rho}\right\|_B^{-1}\right) \tag{3.4}$$

where

$$\left\|\frac{d(x/\|x\|_B)}{d\rho}\right\|_B = \left(\left\|\frac{dx}{d\lambda}\right\|_B^2 \left(\frac{d\lambda}{d\rho}\right)^2 - 1\right)^{1/2}/\rho. \tag{3.5}$$

It is necessary, however, to bound the step-length proposed by (3.4) in cases when the change of the solution is rather small but also the slope $\frac{d\rho}{d\lambda}$ is small. By extrapolation of $\frac{d\rho}{d\lambda}$ as a function of ρ to a zero we obtain from the recent values ρ_{k-1}, ρ_k

$$(\delta\rho_k)_{max} := -\frac{d\rho}{d\lambda}(\rho_k)\,\frac{\frac{d\rho}{d\lambda}(\rho_k) - \frac{d\rho}{d\lambda}(\rho_{k-1})}{\rho_k - \rho_{k-1}} \tag{3.6}$$

where t = 2 is usually chosen corresponding to an expected double zero, although, of course a zero of $\frac{d\rho}{d\lambda}$ need not be forthcoming along the branch.

Utilizing the above ideas the following strategy was used for continuation with the generalized inverse iteration to follow a branch from level ρ_0 to the target level ρ_t.

The Continuation Strategy

Let x_0 be a solution to $\rho = \rho_0$ and let

sg: = signum $(\rho_t - \rho_0)$; k: = 0.

Let RL_k, DX_k, DR_k, k = 0,1, ... and DM_k, k = 1,2, ... be the expressions given by (3.1), (3.2), (3.4) and (3.6), respectively.

```
M1:  p: = if DX_k≠0 then sg•DR_k else ρ_t - ρ_0;
     if k = 0 then goto M2;
     if itold ≤ 2 then p: = p + p;
     q: = if RL_k•RL_k-1 < 0 then 0 else DM_k;
     if sg > 0 and q > 0 then p: = min(p,q);
     if sg < 0 and q < 0 then p: = max(p,q);

M2:  δρ: = sg•min(|p|,|ρ_t-ρ_0|; it: = 0; k: = k+1;

M3:  ρ_k: = ρ_k-1 + δρ;
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An eigenvalue parameter λ is always assumed to be related to the given x by the generalized Rayleigh-quotient (cf.(2.1a)). A total of l_{max} grids is used.

The MG algorithm

Let $x^{(0)}$, $\lambda^{(0)}$ with $\|x^{(0)}\|_0 = \rho^{(0)}$ be a given solution. Set $\rho^{(i)} := R(\rho^{(0)},i)$, $i = 1, \ldots, l_{max}$ and $l := 1$.

1. $k := 1$; smooth (ν_1 steps) $x_0 := \tilde{I}^l_{l-1}(x^{(l-1)})$

   w.r.t.

   $$B^{(l)}x = f^{(l)}(\tilde{x}_0^{(l)})/\lambda^{(l-1)},$$

   result: $x_k^{(l)}$, $\lambda_k^{(l)}$;

2. $j := l$; $r^{(j)} := -f^{(j)}(x_k^{(j)}) + \lambda_k^{(j)}B^{(j)}x_k^{(j)}$;

3. $j := j - 1$; if $j = 0$ then goto 4; $u_0^{(j)} := 0$.

   Smooth (ν_1 steps) $u_j^{(0)}$ w.r.t.

   $$(*)\quad (F^{(j)}(x^{(j)}) - \lambda^{(j)}B^{(j)})u = I^j_{j+1}r^{(j+1)},$$

   result: $u_1^{(j)}$; compute residual $r^{(j)}$ w.r.t. (*)

   and goto 3.

4. Solve

$$\begin{bmatrix} F^{(0)}(x^{(0)}) - \lambda^{(0)}B^{(0)} & -B^{(0)}x^{(0)} \\ -x^{(0)T}B^{(0)} & 0 \end{bmatrix} \begin{bmatrix} \delta^{(0)} \\ * \end{bmatrix} = \begin{bmatrix} I^{0}r^{(1)} \\ 1 \\ 0 \end{bmatrix}$$

$j := 1;$

5. If $j = 1$ then goto 6. Smooth (ν_2 steps)

$u_2^{(j)} := u_1^{(j)} + I_{j-1}^{j}\delta^{(j-1)}$ w.r.t. (*), result $\delta^{(j)}$,

$j := j+1$; goto 5;

6. Smooth (ν_2 steps) $\tilde{x}_k^{(1)} := x_k^{(1)} + I_{1-1}^{1}\delta^{(1-1)}$ w.r.t.

$$B^{(1)}x = f^{(1)}(\tilde{x}_k^{(1)})/\lambda_k^{(1)}$$

and normalize the result to $x_{k+1}^{(1)}$, compute $\lambda_{k+1}^{(1)}$.

If stopping criterion satisfied then

$\lambda^{(1)} := \lambda_{k+1}^{(1)}$; $x^{(1)} := x_{k+1}^{(1)}$; $1 := 1+1$;

if $1 > 1_{max}$ then stop else goto 1

else $k := k+1$; goto 2.

We note that the above algorithm is completely defined if the function R relating the norm levels on the grids is specified and a smoothing method and a stopping criterion have been chosen. This algorithm should have advantages over methods that make use of the parametrization by λ as, for example, is the case for the pseudo-arclength method. There s is

introduced as auxiliary parameter but $\lambda = \lambda(s)$ is kept and hence difficulties have to be expected for a multi-grid version at least in regions where for a given λ not on all grids solutions exist. This will, for example, in general be the case near singular points.

In [1] (cf. also [5]) it was proposed to look for solutions on the finer grid curves on a line orthogonal to the coarse-grid curve and to use additional diagonal shifts of the Jacobians in order to assure that these matrices all have the same number of negative eigenvalues but sacrificing quadratic convergence. While the first strategy is not needed here, once a suitable function R has been chosen, the second was not necessary for the computations reported in 8. For another technique overcoming the first problem in the neighbourhood of simple primary bifurcation points see [15].

We do not analyse here the above MG algorithm theoretically but we present some numerical results which show the efficiency of this approach.

6. Extensions

As mentioned earlier several extensions of the generalized inverse iteration, the continuation strategy and the MG version are possible. We discuss here only a few ideas most of which have been tested numerically.

In order to have a general purpose continuation procedure several features would have to be added to the basic ideas outlined in the preceding sections. In order, for example, to

follow branches bifurcating from the trivial solution at eigenvalues of the linearization it is not possible to follow the trivial solution and branch off since only solutions with non-vanishing norm may be computed. The eigenvalues and -vectors of the linear eigenvalue problems may, however, be computed by the algorithm and be used as starting guesses.

If then along such a primary branch a secondary bifurcation point is detected one may switch to a bifurcating branch by methods as proposed in [6] and implemented in [1]. We have usually preferred to use a simple perturbation.

If it is desired to accurately determine singular points the present approach has the advantage of allowing very crude initial guesses because of its robustness if an appropriate method is used for computation of the singular points. For the determination of (simple) turning points we have, for example, used polynomial interpolation in ρ in order to find extrema of $\lambda(\rho)$. The condition $\frac{d\lambda}{d\rho} = 0$ may, of course, be exploited explicitely. Since in the MG version continuation is done on the coarsest grid a direct method will usually be used for solving the linear systems. Hence the determinant is available for detection and computation of singular points. A point on the curve with $\frac{d\rho}{d\lambda} = 0$ may, of course not be overcome in general by ρ-continuation. In this case a step of λ-continuation should be used as long as $\left|\frac{d\rho}{d\lambda}\right|$ is below a suitable threshold.

We conclude with a remark on the case of unsymmetric but regular B. In this case we define the B-norm as

$$\|x\|_B = \|Bx\|$$

and formally multiply (1.1) from the left by B^T. The modifications necessary in (2.1) are, however, minimal. The Rayleigh-quotient is replaced by $\lambda = f(x)^T Bx/\rho^2$ and the last row of the matrix on the right of (2.1b) becomes

$$[-x_k^T B^T B \quad 0].$$

So one additional matrix-vector product has to be computed. An analogue of the convergence theorem holds and the method has been used successfully, for example, to treat finite-difference discretizations of problems with mixed boundary conditions.

7. Continuation Results

The generalized inverse iteration for ρ-continuation along solution curves of various non-linear eigenvalue problems turned out to be very robust and efficient. By robust we mean that rather large stepsizes were possible without leading to divergence or to jumping to another branch. The efficiency was measured by looking at iteration counts in case the work per iteration was similar or otherwise by comparing computing times. A few results were reported in [12]. The methods that were used for comparisons are the λ-continuation with predictor-step, the pseudo-arclength method of [6], continuation along a suitable component of the solution as

proposed, for example, in [13] and the method of [9]. In the following we restrict the representation to a few examples and a comparison with the pseudo-arclength method as implemented in [1].

Developing an automated continuation algorithm with any underlying method usually makes it necessary to choose smaller stepsizes than the method would allow. This is in particular true for ρ-continuation with the generalized inverse iteration for which rather large steps may be taken (cf.[13]). But then the results would not provide a detailed impression of the solution curve.

The results in this section were obtained with the ρ-continuation strategy of 4. for the basic algorithm of 3. In all cases the nonlinear eigenvalue problems were posed on the unit square $Q = (0,1) \times (0,1)$ with homogeneous Dirichlet boundary conditions. The Laplacian was discretized using the usual five-point difference star. For the finite element method in [1] this was accomplished by choosing the standard triangulation of a square mesh. The right-hand side was discretized pointwise for the ρ-continuation while a suitable quadrature formula is used in [1]. The discretization parameter was $h = 1/4$ since that is a reasonable coarsest grid for a MG algorithm at least for bifurcation from the lowest eigenvalues. Finally, the function g in (3.4) was chosen as

$$g(s) = \frac{1}{4h}\sqrt{s} \ .$$

The first example is the well-known Bratu problem

(7.1) $\quad -\Delta u = \mu \exp(u/(1+\varepsilon u)), \ \varepsilon \geq 0.$

The branch of positive solutions emanating from the origin has one, two or no (simple) turning points. ρ-continuation was started with the constant solution and the branch was followed from $\rho_0 = 1$ to $\rho_t = 100$. Table 7.1 shows the steps taken and the accumulated iteration counts denoted by iter.

ρ	μ	iter
1	1.363	2
12.45	6.670	5
19.88	5.076	9
22.23	4.205	13
26.91	2.622	17
33.03	1.309	21
42.31	.4168	24
58.80	.04711	26
100	.001466	28

Table 7.1 ρ-continuation for Bratu's problem, $\varepsilon = 0$.

Between $\rho = 20$ and $\rho = 40$ the continuation algorithm chooses relatively small steps increasing the total iteration count. We did not try to modify the strategy since it allows to solve efficiently problems with completely different solution curves.

For PLTMGC the accumulated work depended strongly on the μ-steps chosen. Table 7.2 represents the best results we have achieved in a series of runs. The intermediate steps taken by the algorithm are not given.

μ_t	μ	iter
7	7	14
8	7	43
2	2	61
.1	.1	87
.001	.001	102

Table 7.2 PLTMGC - results for Bratu's problem, $\varepsilon = 0$.

Here μ_t denotes the target values used. The turning point is at $\bar{\mu} \approx 7.3$. The algorithm starts in the origin and if $\mu_t > \bar{\mu}$ the algorithm tries to continue to the previous target value but beyond the turning point. Similar results for both methods were obtained for the case $\varepsilon = .2$ when two turning points are present.

We turn now to the simple bifurcation problems

$$-\Delta u = \mu u - u^3, \tag{7.2a}$$

$$-\Delta v = \mu(v - v^3). \tag{7.2b}$$

These problems are equivalent for positive eigenvalue parameters via the transformation $u = \mu^{1/2} v$. The branch bifurcating from the first eigenvalue was computed which for problem (7.2b) does not extend beyond a certain norm-level. Hence this is a test case with small values for $\frac{d\rho}{d\lambda}$ for the

ρ-continuation. Starting solution was the first eigenfunction. The results for both problems are in fact quite similar so we present only those for (7.2a), which was rewritten for the generalized inverse iteration as

$$-\Delta u + u^3 = \mu u.$$

The left-hand side was discretized to yield the vector f in (1.1) while B was the identity matrix.

ρ	μ	iter
1	18.89	1
19.56	63.53	5
22.09	75.33	8
37.34	176.2	11
60	421.3	14

Table 7.3 ρ-continuation for (7.2b), $\rho_0 = 1$, $\rho_t = 60$

μ_t	iter
50	31
200	49

Table 7.4 PLTMGC results for (7.2b)

Many other examples confirmed these results. We make a final remark on the problem

$$-\Delta u = \mu u^p, \quad p > 1 \tag{7.3}$$

which has a solution branch of the form

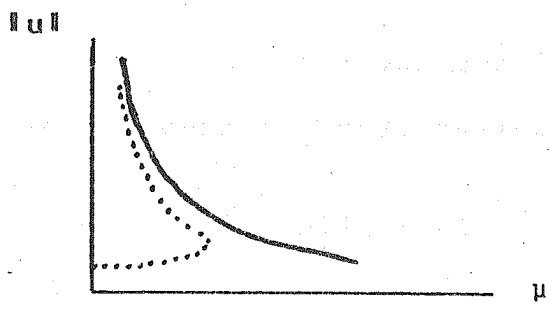

Figure 7.1 Solution branch for (7.3)

No starting guess is easily available and it was suggested in [4] to perturb (7.3) by adding $\delta > 0$ to the right-hand side yielding the dotted curve in order to be able to 'jump' on the branch.

For ρ-continuation a constant initial guess allowed computation of any point on the branch in, for example, 7 steps for $p = 5$. Since, however, all solutions on the branch are proportional, the derivative (3.2) is computed as zero and any other solution is then obtained without a single iteration by simple normalization. There is no similar advantage in following this curve with the pseudo-arclength method.

We have seen that already the strategy proposed in 4. makes ρ-continuation a very competitive method for following branches of finite-dimensional nonlinear eigenvalue problems of the form (1.1).

The numerical results of this section were obtained using single-precision FORTRAN on the IBM 3081 at Arizona State University.

8. Multi-grid Results

In this section some experience will be reported with the MG version of the generalized inverse iteration as given in 5. The program was written for problems of the form

$$-\Delta u = \mu f(u) \quad \text{in } \Omega, \tag{8.1}$$

$$u = g \quad \text{on } \partial\Omega$$

where $\Omega \subset R^2$ was a rather arbitrary domain as allowed in [14] from which auxiliary routines were taken and appropriately modified.

In all computations $\nu_1 = \nu_2 = 2$ was chosen in the MG algorithm. The function R was

$$R(s,i) = s \cdot h^{(0)} / h^{(i)}, \quad i = 1, \ldots, l_{max} \tag{8.2}$$

and the iteration was stopped when

$$|\lambda_{k+1}^{(l)} - \lambda_k^{(l)}| < \text{eps } |\lambda_k^{(l)}|.$$

The smoother used was checkered Gauss-Seidel relaxation, the interpolation $\tilde{I}_{l-1}^{l}$ was of higher order, I_{j-1}^{j} was linear interpolation while the restriction I_{j+1}^{j} was injection. The grids had mesh-widths $h^{(i)} = 2^{-i} h^{(0)}$, $i = 1, \ldots, l_{max}$.

As alternative smoother SSOR-MINRES was used, i.e., an iterate was updated by adding $\alpha_k p_k$ where p_k is the direction

given by a step of the standard SSOR method (and eventually orthogonalized w.r.t. the previous direction) starting from the current iterate. The stepsize $\alpha_k > 0$ was chosen to minimize the residual of the corresponding linear systems. For more indefinite cases this smoothing should be superior. In the computations reported here, where never divergence or no convergence occurred, it only increased the computing time. It was also not necessary to use the normal equations for the smoothing.

Since the computing time is of interest for the MG version we include it in the tables (in seconds). The computations were performed in FORTRAN on the HB 66/80 at the Computing Center of the University of Mainz. We present the results for the MG refinement process at a few selected points on the solution curves of each of the following examples. Q denotes the unit square while K(0,r) is the circle around the origin with radius r. Homogeneous Dirichlet conditions were prescribed in all examples. The starting solution was constant in examples 1 and 3 and the restriction of the eigenfunctions in the other cases.

Example 1 $\qquad -\Delta u = \mu e^u \quad \text{in} \quad Q$

$$h^{(0)} = 1/4, \; l_{max} = 4, \; eps = 10^{-5}$$

Example 2 $\quad -\Delta u = \mu \sin u \quad \text{in} \quad Q$

Branch from eigenvalue $\mu = \lambda_{22} = 8\pi^2$

$h^{(0)} = 1/4,\ l_{max} = 5,\ eps = 10^{-6}$

Example 3 $\quad -\Delta u = \mu(1+u+u^2/2)/(1+u^2/100) \quad \text{in} \quad Q$

$h^{(0)} = 1/4,\ l_{max} = 5,\ eps = 10^{-6}$

Example 4 $\quad -\Delta u = \mu(u-u^3) \quad \text{in} \quad K(0,1/2)$

Branch from eigenvalue $\mu = \lambda_0 = k^2$, where k is twice the smallest zero of the Bessel function J_0.

$h^{(0)} = 1/5,\ l_{max} = 4,\ eps = 10^{-5}$

Following the bifurcating branches in examples 2 and 4 for large values of λ required an increasing number of iterations. The SSOR-MINRES smoother brought considerable improvement but the function R in (8.2) is not appropriate here for large values of λ and would have to be modified suitably if it is desired to compute such solutions.

$\rho^{(0)}$	$\mu^{(3)}$	$k^{(3)}$	time
8	6.142	3	3.731
10	6.637	4	4.281
12	6.806	4	4.288
14	6.653	4	4.471
16	6.179	4	4.607

Table 8.1 Multi-grid results for Example 1

$\rho^{(0)}$	$\mu^{(4)}$	$k^{(4)}$	$\|u\|_\infty$	time
5	79.48066	2	.280538	9.476
15	84.33039	2	.834066	9.363
25	94.83650	3	1.35935	12.381
40	125.3734	3	2.05389	12.826

Table 8.2 Multi-grid results for Example 2

$\rho^{(0)}$	$\mu^{(4)}$	$k^{(4)}$	u(.5,.5)	time
19	8.019515	4	2.10733	13.752
20	8.031423	4	2.22549	13.612
21	8.032892	4	2.34387	13.597
22	8.025606	4	2.46241	13.619

Table 8.3 Multi-grid results for Example 3

$\rho^{(0)}$	$\mu^{(3)}$	$k^{(3)}$	$\|u\|_\infty$	time
1	23.21995	2	.0903495	2.857
4	24.93392	2	.356758	2.735
7	29.50112	3	.603734	3.847
9	35.21668	4	.745784	4.904
11	44.54570	2	.858101	3.792

Table 8.4 Multi-grid results for Example 4

Another problem with the above MG version not present for the continuation with the generalized inverse iteration is encountered near multiple bifurcation points. The eigenvalue λ_{22} in example 2 is simple for the continuous problem but its approximation λ_{22}^h, $h = 1/4$, is not simple. Thus there are some problems for $0 < \rho < 4$. For $h = 1/5$, however, the MG algorithm performs excellent in that range.

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