Title: MG Convergence of Singular Perturbation Problems

Abstract: The Poisson equation in a square is a nice model problem for studying the mg convergence, since it is possible to obtain exact and sharp results. Similarly, one can investigate the singular perturbation problem

$$-\varepsilon \Delta u + \underline{b} \nabla u = f \quad \text{in a square } \Omega,$$

provided that <u>periodic</u> boundary conditions are given. This fact is the basis of the mode analysis.

In the case of the Poisson equation the solutions of the periodic and the Dirichlet boundary value problem are very similar. Not so in the case of singular perturbation problems, where the eigenfunctions related to the Dirichlet condition differs completely from the periodic eigenfunctions. Even in the one-dimensional case $-\varepsilon u''(x)+bu'(x)=f$ there exists no mg convergence proof.

The purpose of this contribution is to close this gap. We show how to prove mg convergence in the Dirichlet case by means of the convergence results for the periodic case.

W. Hackbusch

MULTI-GRID CONVERGENCE FOR A SINGULAR PERTURBATION PROBLEM

Wolfgang Hackbusch
Institut für Informatik und Praktische Mathematik
Christian-Albrechts-Universität
Olshausenstr. 40
D-2300 Kiel 1, Germany

Abstract. An analysis of multi-grid methods applied to singularly perturbed Dirichlet boundary value problems was missing up to now. Only for periodic boundary conditions the Fourier transformation (mode analysis) applies. However, it is not obvious that the convergence results carry over to the Dirichlet case, since the eigenfunctions are quite different in both cases. In this paper we prove a close relationship between multi-grid convergence for the easily analysable case of periodic conditions and the convergence for the Dirichlet case.

1. Introduction

There are results about convergence of multi-grid iterations for quite general problems (cf[2],[6]). However, they do not yield uniform convergence rate with respect to increasing singular perturbation. In such a situation one often studies a pde problem with constant coefficients and periodic boundary conditions. The analysis of this model problem is also named 'local mode analysis' (cf [3],[4]).

For the unperturbed boundary value problems like $-\Delta u=f$ in the unit square the choice of the kind of boundary condition (periodic or Dirichlet) is not essential. The eigenfunctions of the difference schemes are $\exp(i\pi(\nu x+\mu y))$ in the periodic case and

$\sin(\nu\pi x)\sin(\mu\pi x)$ for Dirichlet's condition. The local mode analysis uses the complex Fourier transformation, while the real Fourier transformation is applied in the Dirichlet case. It is not surprising that the close relationship of $\exp(i\pi(\nu x+\mu y))$ and $\sin(\nu\pi x)\sin(\mu\pi x)$ implies almost the same convergence results of the multi-grid iteration applied to the respective discrete equations.

Another situation arises in the case of singular perturbation problems. In this paper we restrict our considerations to the one-dimensional problem

$$(1.1)\qquad -\varepsilon u''(x)+u'(x) = f(x) \quad \text{in } (-1,+1); \qquad u(\pm 1)=0.$$

Replacing the Dirichlet boundary condition by the periodic one, we obtain

$$(1.2)\qquad -\varepsilon u''(x)+u'(x) = f(x) \quad \text{in } (-1,+1); \qquad u(-1)=u(1),\ u'(-1)=u'(1).$$

As again $\exp(i\nu\pi x)$ are the eigenfunctions in the periodic case, the multi-grid iteration can be analysed by means of the Fourier transformation (ie by the local mode analysis). Is the behaviour of the multi-grid convergence in the periodic case (1.2) close to the convergence behaviour in case of (1.1)? A positive answer is not obvious since the eigenfunctions of Eq (1.1) are $e^{-x/2\varepsilon}\sin(\nu\pi(x-1)/2)$. The weighting factor $e^{-x/2\varepsilon}$ reflects the presents of the boundary layer. Also the related scalar products $(u,v)=\int uvdx$ (periodic case) and $(u,v)=\int e^{-x/\varepsilon}uvdx$ are quite different.

Hitherto, an analysis of a multi-grid method for the very simple problem (1.1) has not been given, still less for more complicated problems. The difficulty is caused by the fact that it is not possible to use the eigenfunctions $e^{-x/2\varepsilon}\sin(\ldots)$ and the corresponding weighted scalar product for an analysis of the multi-grid iteration. The first reason is that the exponent $x/2\varepsilon$ of the weighting function becomes h-dependent in the discrete case with step size h. The second reason is that the usual restrictions and prolongations within the multi-grid process are no longer symmetric with respect to the new scalar product. Therefore, the transformation by means of the functions $e^{-x/2\varepsilon}\sin(\ldots)$ does not transform the iteration matrix into a blockdiagonal matrix as in the standard case (cf[5],[13]).

The preceding considerations underline the necessity of studying the convergence of the multi-grid iteration for Eq (1.1), and, in particular, the relation between the convergence behaviour in the respective cases (1.1) and (1.2). Such a result is given in this paper. As a side issue we mention that the presented proving technique is the first one that applies to the lexicographical Gauß-Seidel iteration as smoothing iteration in the Dirichlet case.

The periodic case is studied in § 2. We consider two typical discretizations of the perturbation term u' in Eq (1.1): the backward difference and the centred difference with possibly additional 'artificial viscosity'. The multi-grid iteration is defined in § 2.2. For the backward and centred difference schemes we use different variants of the Gauß-Seidel iteration. The Fourier analysis is recalled in § 2.3. Its results are discussed in § 2.4.

§ 3 is devoted to the Dirichlet boundary condition. In § 3.2 it is proved that a certain norm of the two-grid iteration matrix in the Dirichlet case equals the spectral norm of the iteration matrix in the periodic case plus a matrix of rank 1. This relationship enables the computations of contraction numbers $\ll 1$ for the Dirichlet problem as reported in § 3.3.

2. Analysis in the Periodic Case

2.1 Discretization

Eq (1.1) is replaced by the system

(2.1) $L_h u_h = f_h,$

where h denotes the step size

(2.2) $h = 1/N$ (N even).

The matrix L_h represents a difference scheme $h^{-2}[-a_h \; b_h \; -c_h]$, where

$$(h^{-2}[-a_h \; b_h \; -c_h]\,u_h)(x) = h^{-2}(-a_h u_h(x-h)+b_h u_h(x)-c_h u_h(x+h))$$

for $x=\nu h, \nu\in\mathbb{Z}$. Discretizing the perturbation $u'(x)$ by the backward difference $h^{-1}[-1 \; 1 \; 0]u_h$, we obtain

(2.3$_I$) $a_h=\varepsilon+h, \quad b_h=2\varepsilon+h, \quad c_h=\varepsilon.$

This scheme is known to be stable for all $\varepsilon>0$ and $h>0$. The centred difference $(2h)^{-1}[-1 \; 0 \; 1]u_h\approx u'$ leads us to the scheme

(2.3$_{II}$) $a_h=\varepsilon_h+h/2, \quad b_h=2\varepsilon_h, \quad c_n=\varepsilon_h-h/2$

with wellknown difficulties if ε_h is too small. In particular, (2.3$_{II}$) yields an M-matrix L_h only if the ratio $\varkappa_h=h/\varepsilon_h$ satisfies

(2.4) $\varkappa_h = h/\varepsilon_h \le 2.$

The notation ε_h instead of ε indicates that ε_h might be chosen larger than ε from (1.1). In the latter case it is said that the scheme (2.3$_{II}$) contains 'artificial viscosity'.

Remark 2.1 Scheme (2.3$_I$) can be regarded as special case of (2.3$_{II}$) with $\varepsilon_h:=\varepsilon+h/2$.

This remark does not extend to the two-dimensional problem ..

$$-\varepsilon\Delta u+u_x=f,$$

where the analogues of (2.3$_I$) and (2.3$_{II}$) are

$$h^{-2}\begin{bmatrix} & -\varepsilon & \\ -\varepsilon-h & 4\varepsilon+h & -\varepsilon \\ & -\varepsilon & \end{bmatrix} \text{(backward difference)}, \quad h^{-2}\begin{bmatrix} & -\varepsilon_h & \\ -\frac{h}{2}-\varepsilon_h & 4\varepsilon_h & h/2-\varepsilon_h \\ & -\varepsilon_h & \end{bmatrix} \text{(centred difference)}.$$

2.2. Multi-Grid Iteration

We start with the two-grid iteration. It consists of a 'smoothing step' and 'coarse-grid correction'. The former step requires the choice of a suitable 'smoothing iteration' $\mathcal{S}_h$:

(2.5) $u_h^{old} \mapsto u_h^{new} := \mathcal{S}_h(u_h^{old}, f_h) = S_h u_h^{old} + T_h f_h.$

S_h is the iteration matrix of the iteration $\mathcal{S}_h$. In the cases (2.3_I) and (2.3_{II}) we choose Gauß-Seidel iterations based on different orderings:

(2.6_I) $\mathcal{S}_h$: Gauß-Seidel iteration with lexicographical ordering $(h-1, 2h-1, \ldots, 1-h, 1)$ of the grid points;

(2.6_{II}) $\mathcal{S}_h$: Gauß-Seidel iteration with 'red-black' ordering $(h-1, 3h-1, \ldots, 1-3h, 1-h, 2h-1, 4h-1, \ldots, 1-2h, 1)$.

The coarse-grid correction requires a 'restriction' r from the h-grid onto the 2h-grid and a 'prolongation' p in the opposite direction. The most natural choice is

(2.7) $$(ru_h)(x) = \frac{1}{4}u_h(x-h) + \frac{1}{2}u_h(x) + \frac{1}{4}u_h(x+h), \qquad (x=2\nu h)$$

(2.8) $$(pu_{2h})(x) = \begin{cases} u_{2h}(x) & \text{if } x=2\nu h, \\ [u_{2h}(x-h)+u_{2h}(x+h)]/2 & \text{if } x=(2\nu+1)h. \end{cases}$$

The two-grid iteration is defined by

u_h^j: given j^{th} iterate;

(2.9a) $\bar{u}_h$: result of ν steps of iteration $\mathcal{S}_h$ applied to u_h^j;

(2.9b) $u_h^{j+1} := \bar{u}_h - pL_{2h}^{-1} r(L_h \bar{u}_h - f_h)$.

The 'coarse-grid matrices' $L_{2h} = (2h)^{-1}[-a_{2h} \; b_{2h} \; -c_{2h}]$ corresponding to (2.3_I) and (2.3_{II}), respectively, are

(2.10_I) $a_{2h} = \varepsilon + 2h, \quad b_{2h} = 2\varepsilon + 2h, \quad c_{2h} = \varepsilon,$

(2.10_{II}) $a_{2h} = \varepsilon_{2h} + h, \quad b_{2h} = 2\varepsilon_{2h}, \quad c_{2h} = \varepsilon_{2h} - h$

with ε_{2h} possibly larger than ε_h.

The multi-grid iteration is the iteration (2.9a,b) if we replace the exact solution of $L_{2h}v_{2h} = g_{2h} := r(L_h \bar{u}_h - f_h)$ by few multi-grid iterations in the 2h-grid (cf [6,13]). This approach requires auxiliary step sizes 4h, 8h, ..., too. In case of (2.3_{II}) and (2.10_{II}) one obtains the same ratios $\varkappa_h := h/\varepsilon_h = \varkappa_{2h} := 2h/\varepsilon_{2h}$ at two consecutive levels if ε_h and ε_{2h} are related by

(2.11) $\varepsilon_{2h} = 2\varepsilon_h$.

This choice ensures that $L_{2h}v_{2h}=g_{2h}$ is an equation with the same strength of perturbation as the original problem $L_h u_h = f_h$.

In the following we shall restrict our considerations to the two-grid iteration (2.9a,b), since the multi-grid convergence is closely related to the two-grid convergence (cf [6]).

2.3 Fourier Analysis

Let U_h the set of all grid functions defined on $\{-1,h-1,\dots,1-h\}$. $u_h \in U_h$ may be regarded as a 2-periodically extended function on the infinite grid $h\mathbb{Z}$. Eg we write $u_h(-1-h)$ instead of $u_h(1-h)$. The scalar product on U_h is

$$(u_h,v_h) = \sum_{\nu=0}^{2N-1} u_h(-1+\nu h)\ \overline{v_h(-1+\nu h)}.$$

The constant function with value 1 at all grid points is denoted by $\mathbb{1}$. The space orthogonal to $\mathbb{1}$ is denoted by

$$U_h^{per} = \{v_h:\ (\mathbb{1},v_h)=0\}.$$

Eq (2.1), $L_h u_h = f_h$, can only be solved if $f_h \in U_h^{per}$. Furthermore, the solution of Eq (2.1) becomes unique by the condition $u_h \in U_h^{per}$. The Fourier transformation of $u_h \in U_h$ is $\hat{u}_h$ with

(2.12) $$u_h(x) = \sum_{\mu=1-N}^{N} \hat{u}_h(\mu)\ e^{i\mu\pi x} \qquad (x=\nu h,\ \nu\in\mathbb{Z}).$$

The two-grid iteration (2.9) can be written as

(2.13) $$u_h^{j+1} = M_h u_h^j + N_h f_h$$

where the iteration matrix M_h is

(2.14) $$M_h = C_h S_h^{\nu},\quad C_h = I - pL_{2h}^{-1} r L_h.$$

S_h is the iteration matrix of $\mathcal{S}_h$ (cf(2.5)). L_{2h}^{-1} is defined as mapping from U_{2h}^{per} onto U_{2h}^{per}. For the following analysis it is convenient to extend L_h and L_{2h}^{-1} onto $U_h = U_h^{per} \oplus \mathrm{span}(\mathbb{1})$ by

(2.15) $$L_h\mathbb{1}=\mathbb{1},\quad L_{2h}^{-1}\mathbb{1}=\mathbb{1}.$$

These definitions together with $p\mathbb{1}=\mathbb{1}$, $r\mathbb{1}=\mathbb{1}$ imply $C_h\mathbb{1}=0$.

The frequency μ in (2.12) varies in $\mathbb{Z}_{2N}$ (integers modulo 2N). Define the set of 'low frequencies' by

$$I_{low} = \{1-N/2, 2-N/2, \ldots, N/2\}.$$

Each $\mu\in I_{low}$ can be associated with a 'high frequency' $\mu'=\mu+N\in\mathbb{Z}_{2N}\setminus I_{low}$. It will turn out that the subspace

$$\text{span}\{e^{i\mu\pi x}, e^{i\mu'\pi x}\}, \quad \mu\in I_{low}, \ \mu'=\mu+N$$

is invariant under multiplication by the iteration matrix M_h. Therefore, the Fourier transformation of Eq (2.13) into

$$\hat{u}_h^{j+1} = \hat{M}_h\hat{u}_h^j+\hat{N}_h\hat{f}_h$$

leads us to a blockdiagonal matrix

$$\text{(2.16)} \qquad \hat{M}_h = \text{blockdiag}\{\hat{M}_h^{(\mu)}\}_{\mu\in I_{low}}$$

with 2x2 matrices $M_h^{(\mu)}$. The proof of the representation (2.16) follows from

$$\hat{M}_h = (I-\hat{p}\hat{L}_{2h}^{-1}\hat{r}\hat{L}_h)\hat{S}_h^{\nu}$$

and

$$\text{(2.17)} \qquad \hat{M}_h^{(\mu)} = [I - \hat{p}^{(\mu)}\ (\hat{L}_{2h}^{(\mu)})^{-1}\ \hat{r}^{(\mu)}\ \hat{L}_h^{(\mu)}]\ [\hat{S}_h^{(\mu)}]^{\nu}.$$

The block matrices can be verified to be

$$\text{(2.18)} \qquad \hat{p}^{(\mu)} = \begin{bmatrix}\cos^2(\mu\pi h/2)\\ \sin^2(\mu\pi h/2)\end{bmatrix}, \qquad \hat{r}^{(\mu)} = \begin{bmatrix}\cos^2(\mu\pi h/2) & \sin^2(\mu\pi h/2)\end{bmatrix},$$

$$\text{(2.19)} \qquad \hat{L}_h^{(\mu)} = h^{-2}\text{diag}\{-a_he^{-i\mu\pi h}+b_h-c_he^{i\mu\pi h}, a_he^{-i\mu\pi h}+b_h+c_he^{i\mu\pi h}\} \ (\mu\neq0),$$

$$\text{(2.20)} \qquad \hat{L}_{2h}^{(\mu)} = (2h)^{-2}(-a_{2h}e^{-i\mu2\pi h}+b_{2h}-c_{2h}e^{i\mu2\pi h}) \qquad (\mu\neq0),$$

$$\text{(2.21)} \qquad \hat{L}_h^{(0)} = h^{-2}\text{diag}\{h^2, a_h+b_h+c_h\}, \quad L_{2h}^{(0)}=1 \qquad (\mu=0),$$

The blocks $\hat{S}_h^{(\mu)}$ of the smoothing iterations (2.6_I) and (2.6_{II}) are

$$(2.22_I) \qquad \hat{S}_h^{(\mu)} = \text{diag}\{c_he^{i\mu\pi h}/[b_h-a_he^{-i\mu\pi h}], -c_he^{i\mu\pi h}/[b_h+a_he^{-i\mu\pi h}]\},$$

$$(2.22_{II}) \quad \hat{S}_h^{(\mu)} = \frac{\alpha}{2}\begin{bmatrix}\alpha+1 & \alpha+1\\ \alpha-1 & \alpha-1\end{bmatrix}, \quad \alpha = [c_h e^{i\mu\pi h} + a_h e^{-i\mu\pi h}]/b_h.$$

The representation (2.16) implies

<u>Lemma 2.2</u> $\|M_h\| = \|\hat{M}_h\| = \max_{\mu\in I_{low}} \|\hat{M}_h^{(\mu)}\|$,

where $\|\cdot\|$ is the spectral norm.

The formulae (2.17 - 2.22) show that the 2x2 matrix $\hat{M}_h^{(\mu)}$ depends on the product $\mu\pi h$ only. We rewrite $\hat{M}_h^{(\mu)}$ by $\hat{M}(\mu\pi h)$. Note that $\lim_{\xi\to 0} \hat{M}(\xi) = M_h^{(0)}$ by definition (2.21). Since $\{\mu\pi h: \mu \in I_{low}\} \subset [-\pi/2, +\pi/2]$, one concludes from Lemma 2.2 the estimate stated in

<u>Lemma 2.3</u> $\|M_h\| \le \max_{|\xi|\le\pi/2} \|\hat{M}(\xi)\|$ for all h.

The estimate is sharp because $\|M_h\|$ approaches the right-hand side as $h\to 0$, h/ε=const.

<u>2.4 Results of the Fourier Analysis</u>

We present the results in

<u>Case I:</u> difference schemes (2.3_I), (2.10_I);
smoothing iteration (2.6_I); $\nu=1$

and in

<u>Case II:</u> difference schemes (2.3_{II}), (2.10_{II});
smoothing iteration (2.6_{II}); $\nu=1$.

In Case I define the contraction number

$$\zeta(\frac{h}{\varepsilon}) = \max_{|\xi|\le\pi/2} \|\hat{M}(\xi)\|,$$

since the right-hand side in Lemma 2.3 depends on

$$\varkappa = h/\varepsilon.$$

Some values of $\zeta(\varkappa)$ are shown in Tab. 2.1:

$\varkappa$	0	0.1	0.5	1.0	2.0	10.0	$\to\infty$
$\zeta(\varkappa)$	$\sqrt{2}/3$	0.425	0.348	0.286	0.212	0.069	$\to 0$

Tab. 2.1 Contraction numbers $\zeta(\varkappa)$ for Case I

The worst case is $\varkappa=0$ which is equivalent to the discretization of the unperturbed equation $-u''=f$. Hence, one obtains

Proposition 2.4 In Case I the estimate $\|M_h\| \leq \sqrt{2}/3 \approx 0.47$ holds for all h and all $\varepsilon > 0$.

Proposition 2.4 implies uniform two-grid convergence for all h and ε. Of course the contraction number $\|M_h\|$ is a too pessimistic measure of the convergence. In fact the estimate $\rho(M_h) \leq 1/3$ can be proved for the convergence rate.

The reason of $\zeta(\varkappa) \to 0$ $(\varkappa \to \infty)$ is the fact that the larger $\varkappa$ is the better the Gauß-Seidel iteration (2.6_I) works. Especially for $\varkappa = \infty$ (ie $\varepsilon = 0$) one step of the Gauß-Seidel iteration yields the exact result. This would not be true if we invert the ordering of the grid point. Applying the Gauß-Seidel iteration from the right to the left, we would obtain a very poor or even divergent two-grid iteration. This fact has severe consequences for the two-dimensional case. The Gauß-Seidel iteration $\mathcal{S}_h$ has to follow the flow direction (cf [3]). This is often inconvenient and one would prefer a multi-grid method admitting Gauß-Seidel iterations of any ordering, eg of the neutral choice in (2.6_{II}).

In Case II with $\varepsilon_{2h} = 2\varepsilon_h$ the contraction numbers $\zeta(\varkappa_h)$, $\varkappa_h = h/\varepsilon_h = \varkappa_{2h} = 2h/\varepsilon_{2h}$, are given in Tab. 2.2 :

$\varkappa_h = \varkappa_{2h}$	2.0	1.5	1.14	1.0	0.8	0.7	0.6	0.5	0.1	$\to 0$
$\zeta(\varkappa_h)$	$\sqrt{2}^*$	$15/16^*$	.704	.662	.630	.623	.622	.626	.684	$\to 1/\sqrt{2}$

Tab. 2.2 Contraction numbers $\zeta(\varkappa_h)$ for Case II with $\varepsilon_{2h} = 2\varepsilon_h$

Although $\varkappa_h = 2$ yields a reasonable difference scheme (cf (2.4)), not only the contraction number but also the spectral radius is ≥ 1 for $\varkappa_h \geq 2$. The divergence is not the result of the 'wrong' choice $\varepsilon_{2h} = 2\varepsilon_h$. All values in Tab. 2.2 marked by an asterisk cannot be improved by any choice of ε_{2h}. These values are $\|\hat{M}(\pi/2)\| = \varkappa_h \sqrt{1+\varkappa_h^2/4}\,/2$ which are obviously independent of $\varkappa_{2h}$ or ε_{2h}. The choice $\varepsilon_{2h} = \varepsilon_h$ (instead of $\varepsilon_{2h} = 2\varepsilon_h$) does improve the other contraction numbers of Tab 2.2 as can be seen from Tab 2.3.

$\varepsilon_h=\varepsilon_{2h}/2$	2.0	1.5	1.14	1.0	0.8	0.7	0.6	0.5	0.1	$\to 0$
$\rho(\varkappa_h)$	$\sqrt{2}$	15/16	.656	.565	.484	.456	.435	.418	.386	$\to 0.385$

Tab. 2.3 Contraction numbers $\rho(\varkappa_h)$ for Case II with $\varepsilon_{2h}=\varepsilon_h$

These numbers clearly indicate the strategy of choosing ε_{2h} (and $\varepsilon_{4h}, \varepsilon_{8h}, \ldots$ in the multi-grid case). If $\varkappa_h \geq 0.6$ the choice $\varepsilon_{2h}=\varepsilon_h$ implying $\varkappa_{2h} \geq 1.2$ would cause a too poor convergence of the multi-grid process in the 2h-grid. In that case $\varkappa_{2h}=\varkappa_h$ (ie $\varepsilon_{2h}=2\varepsilon_h$) is required. Only if $\varkappa_h < 0.6$ we may choose ε_{2h} closer to ε_h, since then the convergence in the h-grid is improved while $\varkappa_{2h} \leq 0.6$ guarantee the applicability of the multi-grid iteration in the coarser grid. The resulting definition of ε_{2h} is

(2.23) $\qquad \varepsilon_{2h}=\max(\varepsilon_h, 2h/0.6) \approx \max(\varepsilon_h, 3.3\,h).$

Using this definition also for $\varepsilon_{4h}, \varepsilon_{8h}, \ldots$ we can ensure $\varkappa_{2^i h} \leq \max(0.6, \varkappa_h)$ for $i=1,2,\ldots$

<u>Proposition 2.5</u> In Case II with $\varepsilon_h \leq \varepsilon_{2h} \leq 2\varepsilon_h$ the estimate $\|M_h\| \leq 1/\sqrt{2}$ holds for all $\varkappa_h=h/\varepsilon_h \leq 1.148$.

3. Analysis in the Dirichlet Case

3.1 Discretization and Notations

The Dirichlet problem (1.1) is discretized by the same difference schemes as in Section 2.1. The solution is in

$$U_h^o = \{v_h : v_h(\pm 1) = 0\} \subset U_h.$$

We shall extensively use the fact that the orthogonal projection

(3.1) $$Q_h : U_h \to U_h^{per}, \quad Q_h u_h := u_h - \frac{1}{2N}(\mathbb{1}, u_h)\mathbb{1} \quad \perp \mathbb{1}$$

is a one-to-one mapping from U_h^o onto U_h^{per}.

From now the symbols

$$S_h, \ C_h, \ M_h \qquad \text{(cf (2.6), (2.14))}$$

denote the respective matrices for the Dirichlet boundary condition, whereas the matrices in the periodic case are renamed

$$S_h^{per}, \ C_h^{per}, \ M_h^{per}.$$

3.2 Relationship of M_h and M_h^{per}

As $S_h : U_h^o \to U_h^o$ and $S_h^{per} : U_h^{per} \to U_h$, one has to compare $Q_h S_h : U_h^o \to U_h^{per}$ and $Q_h S_h^{per} Q_h : U_h^o \to U_h^{per}$. The following considerations are based on

<u>Lemma 3.1</u> In Case I (cf § 2.4) the identity

(3.2) $$Q_h S_h = Q_h S_h^{per} Q_h + Q_h w_h \cdot \phi_h^T$$

holds, where the vectors w_h and ϕ_h are given by

(3.3a) $$w_h(-1+\nu h) = (a_h/b_h)^{\nu} = \left(\frac{\varepsilon+h}{2\varepsilon+h}\right)^{\nu} \qquad (0 \le \nu \le 2N-1),$$

(3.3b) $$\phi_h = [I - (S_h^{per})^T] e_o \in U_h^{per} \text{ with } e_o(\pm 1) = 1, \ e_o(x) = 0 \text{ otherwise.}$$

<u>Proof.</u> Set $v_h = S_h u_h \in U_h^o$ with $v_h(\pm 1) := 0$. By definition of the Gauß-Seidel iteration (2.6_I) the function v_h fulfils

(3.4) $$v_h(x) = [a_h v_h(x-h) + c_h u_h(x+h)]/b_h \qquad (x = h-1, 2h-1, \ldots, 1-h).$$

Set

$$u_h^{per} := Q_h u_h; \quad v_h^{per} := S_h^{per} u_h^{per}; \quad \delta u_h := u_h^{per} - u_h = \alpha \mathbb{1}$$

with $\alpha = (\mathbb{1}, u_h)/2N$. By definition v_h^{per} satisfies

(3.5) $\quad v_h^{per}(x) = [a_h v_h^{per}(x-h) + c_h u_h^{per}(x+h)]/b_h \qquad (x=h-1, 2h-1, \ldots, 1-h, 1)$

By (3.4) and (3.5) the difference

$$\delta v_h := v_h^{per} - v_h$$

fulfils

(3.6) $\quad \delta v_h(x) = [a_h \delta v_h(x-h) + c_h \delta u_h(x+h)]/b_h \qquad (x=h-1, 2h-1, \ldots, 1-h)$

As $a_h + c_h = b_h$, the constant function $\delta u_h = \alpha \mathbb{1}$ is a solution of

(3.7) $\quad \delta u_h(x) = -[a_h \delta u_h(x-h) + c_h \delta u_h(x+h)]/b_h \qquad (x=h-1, 2h-1, \ldots, 1-h, 1).$

Set $z_h := \delta u_h - \delta v_h$ and subtract (3.7) from (3.6):

$$z_h(x) = [a_h z_h(x-h) + 0]/b_h = (a_h/b_h) z_h(x-h) \qquad (x=h-1, \ldots, 1-h).$$

Hence z_h has the representation $z_h(-1+\nu h) = (a_h/b_h)^{\nu} z_h(-1)$ or

$$z_h = z_h(-1)\, w_h$$

with w_h from (3.3a). Since $Q_h \delta u_h = \alpha Q_h \mathbb{1} = 0$, we have

$$Q_h S_h u_h = Q_h v_h = Q_h(v_h^{per} - \delta v_h) = Q_h(S_h^{per} u_h^{per} - \delta u_h + z_h) =$$

$$Q_h S_h^{per} Q_h u_h + Q_h z_h = Q_h S_h^{per} Q_h u_h + z_h(-1)\, Q_h w_h.$$

From $u_h(-1) = v_h(-1) = 0$ we conclude

$$z_n(-1) = \delta u_h(-1) - \delta v_h(-1) = [u_h^{per}(-1) - u_h(-1)] - [v_h^{per}(-1) - v_h(-1)] =$$

$$= u_h^{per}(-1) - v_h^{per}(-1) = \{[I - S_h^{per}] Q_h u_h\}(-1) =$$

$$= (e_o, [I - S_h^{per}] Q_h u_h) = (\phi_h, u_h)$$

with e_o and $\phi_h = Q_h \phi_h$ from (3.3b). ∎

A similar result can be obtained in Case II:

<u>Lemma 3.2</u> In Case II (cf § 2.4) identity (3.2) holds with

(3.8a) $\quad w_h(\pm 1) = 1, \; w_h(x) = 0$ otherwise;

(3.8b) $$\phi_h(x) = \begin{cases} (a_h^2 + c_h^2)/b_h^2 & \text{if } x = \pm 1 \\ -c_h^2/b_h^2 & \text{if } x = -1+2h \\ -a_h^2/b_h^2 & \text{if } x = 1-2h \\ 0 & \text{elsewhere} \end{cases} \in U_h^{per}.$$

Proof. Apply Q_hS_h and $Q_hS_h^{per}Q_h$ to the unit vector e_i with $e_i(-1+ih)=1$, $e_i(x)=0$ otherwise. If i is odd the first half-step of the Gauß-Seidel iteration (2.6_{II}) annihilates the non-zero coefficient: $Q_hS_he_i=Q_hS_h^{per}Q_he_i=0$, proving $\phi_h(-1+ih)=0$ for odd i. If i is even and in the interval $[-1+4h,1-4h]$, $Q_hS_he_i$ and $Q_hS_h^{per}Q_he_i$ equal the vector

$$(3.9) \qquad Q_h(\ldots,\, 0,\, c_h^2/b_h^2,\, c_h/b_h,\, \underset{\substack{\uparrow\\ x=-1+ih}}{2a_hc_h/b_h^2},\, a_h/b_h,\, a_h^2/b_h^2,\, 0,\ldots)^T,$$

demonstrating $\phi_h(-1+ih)=0$. It remains to discuss the cases $i=2$ and $i=2N-2$. Let $i=2$. $Q_hS_h^{per}Q_he_2$ is again the vector (3.9), whereas $Q_hS_he_2$ has a zero at $x=-1$ instead of c_h^2/b_h^2. Hence

$$Q_hS_he_2=Q_hS_h^{per}Q_he_2-(c_h^2/b_h^2)Q_hw_h=Q_hS_h^{per}Q_he_2+Q_hw_h\phi_h^Te_2.$$

The case $i=2N-2$ (ie $x=1-2h$) is analogous. The definition of $\phi_h(\pm 1)$ is irrelevant since ϕ_h is multiplied by some $u_h \in U_h^o$ with $u_h(\pm 1)=0$. The choice of $\phi_h(\pm 1)$ is uniquely determined by $\phi_h \in U_h^{per}$. ∎

We recall the definition of the coarse-grid matrices C_h and C_h^{per} (cf (2.14)):

$$C_h = I-pL_{2h}^{-1}rL_h \qquad \text{in } U_h^o,$$
$$C_h^{per} = I-p^{per}\,(L_{2h}^{per})^{-1}r^{per}\,L_h^{per} \qquad \text{in } U_h^{per}.$$

Lemma 3.3 $Q_hC_h=C_h^{per}Q_h$ in Case I and Case II.

Proof. Let $u_h \in U_h^o$ be arbitrary. By $L_h^{per}\mathbb{1}=0$ one has

$$(L_hu_h)(x) = (L_h^{per}u_h)(x) = (L_h^{per}Q_hu_h)(x) \quad \text{for } x=h-1,2h-1,\ldots,1-h.$$

Hence the restrictions r and r^{per} yield $d_{2h}:=rL_hu_h$ and $d_{2h}^{per}:= r^{per}L_h^{per}Q_hu_h$ with

$$d_{2h}(x) = d_{2h}^{per}(x) \quad \text{for } x=2h-1,4h-1,\ldots,1-4h,1-2h.$$

Since the values $d_{2h}(\pm 1)$ play no rôle in the Dirichlet case, we may identify d_{2h} and $d_{2h}^{per} \in U_{2h}^{per}$. The coarse-grid solutions

$v_{2h}=L_{2h}^{-1}\,d_{2h} \in U_{2h}^o$ and $v_{2h}^{per}=(L_{2h}^{per})^{-1}d_{2h} \in U_h^{per}$ differ by a multiple of $\mathbb{1}$. Thus,

$$v_{2h}^{per} = Q_{2h}v_{2h}$$

follows. Set $v_h := pv_{2h}$ and $v_h^{per} := p^{per}v_{2h}^{per}$ and note that $p^{per}: U_{2h}^{per} \to U_h^{per}$ and $p^{per}\mathbb{1} = \mathbb{1}$ imply $p^{per}Q_{2h} = Q_h p^{per}$:

$$v_h^{per} := p^{per}v_{2h}^{per} = p^{per}Q_{2h}v_{2h} = Q_h p^{per}v_{2h} = Q_h p v_{2h},$$

where the last step uses $p = p^{per}$ on U_{2h}^{o}. Hence, we have

$$Q_h p L_{2h}^{-1} r L_h u_h = v_h^{per} = p^{per}(L_{2h}^{per})^{-1} r^{per} L_h^{per} Q_h u_h$$

proving $Q_h p L_{2h}^{-1} r L_h = p^{per}(L_{2h}^{per})^{-1} r^{per} L_h^{per} Q_h$ and $Q_h C_h = C_h^{per} Q_h$. ∎

Combining Lemmata 3.1 to 3.3 and recalling $M_h = C_h S_h^{\nu}$ (cf (2.14)), one obtains for $\nu=1$ the relation

$$Q_h M_h = Q_h C_h S_h = C_h^{per} Q_h S_h = C_h^{per} [Q_h S_h^{per} Q_h + Q_h w_h \cdot \phi_h^T] =$$
$$= M_h^{per} Q_h + C_h^{per} Q_h w_h \cdot \phi_h^T.$$

Hence, we have proved

Proposition 3.4 Assume Case I or Case II with $\nu=1$. Then the two-grid iteration matrices M_h (associated with Dirichlet condition) and M_h^{per} (associated with periodic condition) are related by

$$\text{(3.10)} \qquad Q_h M_h = M_h^{per} Q_h + C_h^{per} Q_h w_h \cdot \phi_h^T$$

with w_h and $\phi_h \in U_h^{per}$ from (3.3a,b) (Case I) or (3.8a,b) (Case II), respectively.

In the introduction we raised the question: What is a suitable norm (or scalar product) in the Dirichlet case? The approach of this chapter leads us to the choice of

$$|||u_h||| := \|Q_h u_h\| = (Q_h u_h, Q_h u_h)^{1/2},$$

which is obviously a norm on U_h^{o}. The associated matrix norm is denoted by $|||\cdot|||$, too.

$$|||A||| := \sup\{|||Au_h||| / |||u_h||| : 0 \neq u_h \in U_h^{o}\}.$$

<u>Proposition 3.5</u> Under the assumptions of Proposition 3.4 the contraction number of the two-grid iteration in the Dirichlet case equals

(3.11) $$|||M_h||| = \|M_h^{per}+C_h^{per}Q_h w_h \cdot \phi_h^T\|.$$

<u>Proof.</u> For any $u_h \in U_h^o$ the estimate

$$|||M_h u_h||| = \|Q_h M_h u_h\| = \|[M_h^{per}Q_h + C_h^{per}Q_h w_h \cdot \phi_h^T] u_h\| = \|[M_h^{per}+C_h^{per}Q_h w_h \cdot \phi_h^T]Q_h u_h\| \leq$$

$$\leq \|M_h^{per}+C_h^{per}Q_h w_h \cdot \phi_h^T\| \, \|Q_h u_h\| = \|M_h^{per}+C_h^{per}Q_h w_h \cdot \phi_h^T\| \, |||u_h|||$$

is valid. Further, the equality sign holds for an appropriate $u_h \in U_h^o$. ∎

The right-hand side in Eq (3.11) can be estimated by means of

<u>Lemma 3.6</u> $\|A+x \cdot y^T\|^2 \leq \|A\|^2 + \delta$ with

$$\delta = \left| \frac{\|x\|^2\|y\|^2}{2} + (Ay,x) \right| + \left(\frac{\|x\|^4\|y\|^4}{4} + \|x\|^2\|y\|^2(Ay,x) + \|x\|^2\|Ay\|^2 \right)^{1/2}.$$

<u>Proof.</u> Use $\|A+xy^T\|^2 = \|(A^T+yx^T)(A+xy^T)\| = \|A^TA+yx^TA+\|x\|^2yy^T+A^Txy^T\|$. The estimate

$$\|A+xy^T\|^2 \leq \|A\|^2 + \|yx^TA+\|x\|^2yy^T+A^Txy^T\|$$

yields the inequality of Lemma 3.6. ∎

Setting

(3.12) $$A=M_h^{per}, \quad x=C_h^{per}Q_h w_h, \quad y=\phi_h$$

we can apply Lemma 3.6 to the right-hand side of Eq (3.11). Note that $\|A\|^2 = \|M_h^{per}\|^2$ is known from § 2, while δ can immediately evaluated, since x and y are explicitly known. δ depends on N=1/h and $\varkappa = h/\varepsilon$ (or $\varkappa_h = h/\varepsilon_h$). A first necessary requirement for a uniform estimate $|||M_h||| < 1$ is the uniform boundedness of δ for all N.

Proposition 3.7 In Case I and in Case II ($\varkappa_h=h/\varepsilon_h$ constant) the number δ evaluated for (3.12) is uniformly bounded for all $N=1/h$ (and for all $\varkappa=h/\varepsilon$ in Case I).

Proof. It is sufficient to prove $\|x\|, \|y\| \leq \text{const} < \infty$ for all N. In Case II the definitions (3.8a,b) imply $\|w_h\|=1$, $\|\phi_h\|\leq 2$ from which $\|x\|,\|y\|\leq \text{const}$ follows. In Case I the estimate $\|y\|=\|\phi_h\|\leq 2$ is obvious, whereas $\|w_h\|^2 \leq (2+\varkappa)^2/(3+2\varkappa)$ is not uniform with respect to $\varkappa=h/\varepsilon$. However, $\|x\| = \|C_h^{per} Q_h w_h\|$ is not increasing to infinity as $\varkappa \to \infty$, since for $\varkappa \to \infty$ the functions w_h becomes smoother and the coarse-grid correction C_h^{per} is increasingly efficient. A longer exercise shows

$$\|x\|^2 = \|C_h^{per} Q_h w_h\|^2$$

$$= \frac{1-\alpha^{4N}}{1-\alpha^4}\left[(1-\beta)^2 + \left(\alpha-\beta\frac{\alpha^2+1}{2}\right)^2\right] + \frac{1-\gamma^{2N}}{1-\gamma^2}\left[1+\left(\frac{1+\gamma}{2}\right)^2\right]\delta^2$$

$$-2\frac{1-\alpha^{2N}\gamma^N}{1-\alpha^2\gamma}\left[(1-\beta)+\left(\alpha-\beta\frac{\alpha^2+1}{2}\right)\frac{1+\gamma}{2}\right]\delta + (2\Delta+\eta)\eta$$

with

$$\alpha = a_h/b_h,\ \beta=\alpha^2(2a_h+c_h)/(a_{2h}-\alpha^2 c_{2h}),\ \gamma=a_{2h}/c_{2h},$$

$$\delta=(\beta-1)(1-\alpha^{2N})/(\gamma^N-1),\ \Delta=\alpha^{2N-2}\left(\alpha-\beta\frac{\alpha^2+1}{2}\right)-\gamma^{N-1}\delta\frac{1+\gamma}{2},\ \eta=(\alpha^{2N}-1)/2$$

from which $\|x\|\leq \text{const}$ can be concluded for all $\varkappa$. ∎

In the next subsection the number δ from Lemma 3.7 is evaluated for various parameters.

3.3 Contraction Numbers in the Dirichlet Case

Tab 3.1 contains the number δ from Lemma 3.6 in Case I. More precisely, the values in Tab 3.1 are the maximal δ's taken over $h\in\{1/8,1/16,1/32,1/64\}$. It turns out that the dependence of δ on $h=1/N$ is rather noticeable. From these values δ and from the bounds of $\|M_h^{per}\|$ given in Tab 2.1 one obtains the bounds of $|||M_h|||$ according to Lemma 3.6.

$\varkappa$	0	0.1	0.5	1.0	2.0	10.0	$\to\infty$
δ	0.64	0.537	0.382	0.323	0.277	0.240	$\to 0$
$\|\|\|M_h\|\|\|\le$	0.93	0.847	0.709	0.636	0.567	0.494	$\to 0$

Tab 3.1 Values of δ (cf Lemma 3.6) and bounds of $|||M_h|||$ in Case I

The figures show the counterpart of Proposition 2.4 in the case of Dirichlet boundary conditions:

Proposition 3.8 In Case I with Dirichlet boundary condition the estimate

(3.13) $\quad |||M_h||| \le c<1$

holds independent of h and $\varepsilon>0$.

Inequality (3.13) proves uniform convergence of the two-grid iteration, although the bound c=0.93 suggested by Tab 3.1 is too pessimistic.

$\varkappa_h=\varkappa_{2h}$	2.0	1.5	1.14	1.0	0.7	0.5	0.1	$\longrightarrow 0$
δ	0.89	0.51	0.35	0.31	0.25	0.23	0.200	$\longrightarrow$ 0.199
$\|\|\|M_h\|\|\|\le$	1.7	1.18	0.92	0.87	0.80	0.79	0.817	$\longrightarrow$ 0.836

Tab. 3.2 Values of δ and upper bounds of $|||M_h|||$ in Case II

The respective numbers in Case II are exhibited in Tab 3.2. Convergence follows for all $\varkappa_h\le 1.148$ which are also considered in Proposition 2.5:

Proposition 3.9 Consider Case II with $\varepsilon_{2h}=2\varepsilon_h$ for the Dirichlet problem (1.1). The estimate (3.13) is valid for all h and all $\varkappa_h=h/\varepsilon_h\leq 1.148$.

Conclusion. Even the rough estimate of $|||M_h|||$ by means of Lemma 3.5 was sufficient for proving that the contraction member $\|M_h^{per}\|$ and $|||M_h|||$ are closely related.

References.

[1] BÖRGERS, C.: Mehrgitterverfahren für eine Mehrstellendiskretisierung der Poissongleichung und für eine zweidimensionale singulär gestörte Aufgabe. Diplomarbeit (master thesis), Universität Bonn 1981

[2] BRAESS, D. and W. HACKBUSCH: A new convergence proof for the multigrid method including the V-cycle. To appear in SIAM J Numer Anal

[3] BRANDT, A.: Multi-level adaptive techniques (MLAT) for singular-perturbation problems. In: [12] 53-142

[4] BRANDT, A.: Guide to multigrid development. In: [7] 220-312

[5] HACKBUSCH, W.: On the multi-grid method applied to difference equations. Computing 20 (1978) 291-306

[6] HACKBUSCH, W.: Multi-grid convergence theory. In: [7] 177-219

[7] HACKBUSCH, W. and U. TROTTENBERG (eds): Multi-grid methods. Proceedings, Köln-Porz, Nov 1981. Lecture Notes in Maths 960. Springer, Berlin 1982

[8] HEMKER, P. W.: An accurate method without directional bias for the numerical solution of a 2-D elliptic singular perturbation problem. to appear

[9] HEMKER, P. W.: Mixed defect correction iteration for the accurate solution of the convection diffusion equation. In: [7] 485-501

[10] HEMKER, P. W.: Numerical aspects of singular perturbation problems. Preprint NW 133/82, Mathematisch Centrum, Amsterdam 1982

[11] HEMKER, P. W. and P. M. DE ZEEUW: Defect correction for the solution of a singular perturbation problem. Preprint NW 132/82, Mathematisch Centrum, Amsterdam 1982

[12] HEMKER, P. W. and J. J. H. MILLER (eds): Numerical analysis of singular perturbation problems. Proceedings, Nijmegen, May-June 1978. Acad Press, London 1979

[13] STÜBEN, K. and U. TROTTENBERG: Multigrid methods: fundamental algorithms, model problem analysis and applications. In: [7] 1-176

[14] VAN ASSELT, E. J.: The multi grid method and artificial viscosity. In: [7] 313-326