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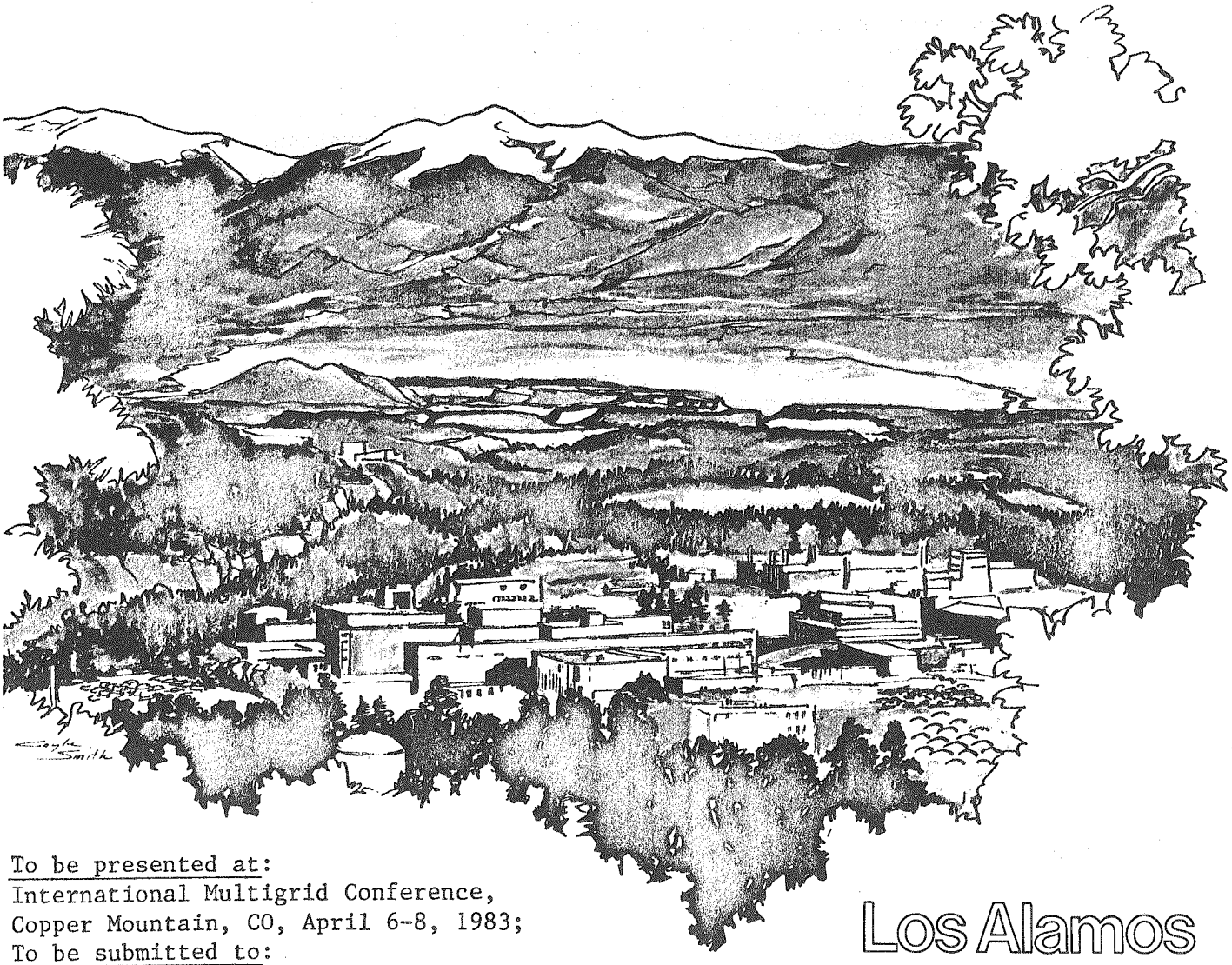

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Abstract

One major problem with the multigrid method has been that each new grid configuration has required a major programming effort to develop a code that specifically handles that grid configuration. Such a penalty is not required for methods like SOR, ICCG, etc.; in these methods, one need only specify the matrix problem no matter what the grid configuration. In previous work, this defect was overcome and a code written in which one needed only to specify the logically rectangular, positive definite matrix problem; the code did everything else necessary to set up the auxilliary coarser problem to achieve a multigrid solution. In this paper, the previous work is extended to a single nonsymmetric equation.

I. REVIEW OF THE POSITIVE DEFINITE CASE

Let us begin with a summary of the previous work for positive definite, logically rectangular problems. Let the continuous problem

$$LU = F$$

be approximated on a grid G^M by

$$L^M U^M = F^M \quad . \tag{1.1}$$

In the simplest form of the multigrid method, one constructs a sequence of

grids $G^1, \ldots, G^M$ with corresponding mesh sizes $h_1, \ldots, h_M$, where $h_{i-1} = 2h_i$. One does a fixed number, IM, of relaxation sweeps (Gauss-Seidel, for example) on (1.1) and then drops down to grid G^{M-1} and the equation

$$L^{M-1}V^{M-1} = f^{M-1} \equiv I_M^{M-1}(F^M - L^M v^M) \quad , \tag{1.2}$$

where V^{M-1} is to be the G^{M-1} approximation to $V^M \equiv U^M - u^M$, where $v^M = u^M$ is the last iterate on G^M and where $I_M^{M-1} : G^M \to G^{M-1}$. To solve (1.2) approximately, one resorts to recursion, taking ID relaxation sweeps on G^k before dropping down to G^{k-1}, $M-1 \geq k \geq 2$ and the equation

$$L^{k-1}V^{k-1} = f^{k-1} \equiv I_k^{k-1}(f^k - L^k v^k) \quad .$$

When grid G^1 is reached, the equation $L^1V^1 = f^1$ can be solved directly and $v^2 \leftarrow v^2 + I_1^2 v^1$ performed. Then one does IU relaxation sweeps on G^{k-1} before forming $v^k \leftarrow v^k + I_{k-1}^k v^{k-1}$, $3 \leq k \leq M$.

If L is positive definite with well behaved coefficients, one can take I_{k-1}^k to be bilinear interpolation and I_k^{k-1} to be $(I_{k-1}^k)^*$ or just injection. If, however, L has poorly behaved coefficients (e.g., $LU = -\nabla\cdot(D\nabla U)$, where D is piecewise constant and jumps by orders of magnitude across interfaces), then a more exotic choice of these operators is required.[D] The idea is to use the difference equation to define the interpolation $I_{k-1}^k : G^{k-1} \to G^k$. If $(IF,JF) \in G^k$ is the same point as $(IC,JC) \in G^{k-1}$, I_{k-1}^k is just given by replacement:

$$(I_{k-1}^k v^{k-1})_{IF,JF} = v_{IC,JC}^{k-1} \quad .$$

If $(IF+1,JF) \in G^k$ lies between $(IC,JC) \in G^{k-1}$ and $(IC+1,JC) \in G^{k-1}$ and L^k

has at (IF+1,JF) the template

$$\begin{bmatrix} -NW & -N & -NE \\ -W & C & -E \\ -SW & -S & -SE \end{bmatrix}, \tag{1.3}$$

then

$$(I_{k-1}^{k} v^{k-1})_{IF+1,JF} = \frac{(NW+W+SW)v_{IC,JC}^{k-1} + (NE+E+SE)v_{IC+1,JC}^{k-1}}{C-N-S} . \tag{1.4}$$

The idea is to sum the columns of (1.3) to average out the vertical dependence. A similar formula holds for $(IF,JF+1) \varepsilon G^k$ between $(IC,JC) \varepsilon G^{k-1}$ and $(IC,JC+1) \varepsilon G^{k-1}$. Enough information is now present to use the difference template at points $(IF+1,JF+1) \varepsilon G^k$ which are the centers of rectangles formed by (IC,JC), (IC+1,JC), (IC,JC+1), $(IC+1,JC+1) \varepsilon G^{k-1}$. That is, if (1.3) now represents the template of such a point (IF+1,JF+1), then

$$\begin{aligned} (I_{k-1}^{k} v^{k-1})_{IF+1,JF+1} = (&W(I_{k-1}^{k} v^{k-1})_{IF,JF+1} + E(I_{k-1}^{k} v^{k-1})_{IF+2,JF+1} \\ &+ S(I_{k-1}^{k} v^{k-1})_{IF,JF} + N(I_{k-1}^{k} v^{k-1})_{IF,JF+2} \\ &+ SW(v_{IF,JF}^{k-1}) + SE(v_{IF+2,JF}^{k-1}) + NW(v_{IF,JF+2}^{k-1}) \\ &+ NE(v_{IF+2,JF+2}^{k-1}))/C . \end{aligned}$$

To complete the description in the positive definite case, one takes $I_k^{k-1} = (I_{k-1}^k)^*$ and $L^{k-1} = I_k^{k-1} L^k I_{k-1}^k$.

The original motivation for black box multigrid was to extend multigrid to handle operators like $-\nabla\cdot(D\nabla\cdot)$ where D jumps by orders of magnitude across internal interfaces. The result was a method in which

only the matrix of the finest grid equations needs to be specified. An additional bonus is that the number of unknowns on the finest grid do not have to be of a special form (e.g., number of x points = NXM = NXO(2^k) + 3, etc.) in order that special cases in interpolation be avoided; in fact, as shown in [D], even equations on curved domains can be solved. [D] also exhibited success with mapping techniques and Lagrangian grids. This paper will exhibit success for all of the above for the case of a single nonsymmetric equation.

II. THE NONSYMMETRIC CASE

As a model problem for the nonsymmetric case, consider

$$L_\varepsilon U \equiv -\varepsilon \Delta U + \frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} = F \quad , \text{ on } \Omega \subset R^2 \quad , \tag{2.1}$$

where $\varepsilon > 0$, and the corresponding discretization on a uniform grid with mesh spacing h:

$$L_\varepsilon^h U_{i,j} = -\varepsilon \Delta_h U + D_0^{x,h} U_{i,j} + D_0^{y,h} U_{i,j} = F_{i,j} \quad , \tag{2.2}$$

where

$$\Delta_h U_{i,j} = \frac{1}{h^2}(-4U_{i,j} + U_{i-1,j} + U_{i+1,j} + U_{i,j-1} + U_{i,j+1}) \quad ,$$

$$D_0^{x,h} U_{i,j} = \frac{1}{2h}(U_{i+1,j} - U_{i-1,j}) \quad ,$$

and

$$D_0^{y,h} U_{i,j} = \frac{1}{2h}(U_{i,j+1} - U_{i,j-1}) \quad .$$

The behavior of (2.2) depends a lot on boundary data. If, for example, $\Omega = (0,1) \times (0,1)$ and U is specified on all the points on the boundary of Ω, then if $\varepsilon \ll \frac{1}{2}h$, the solution of (2.2) has oscillations which have nothing to do with the solution of (2.1). If however, U is specified only on $(x,0)$, $0 \le x \le 1$, and $(0,y)$, $0 \le y \le 1$, and proper care is taken at the boundaries $(x,1)$, $0 \le x \le 1$ and $(1,y)$, $0 \le y \le 1$, then (2.2) can yield a reasonable approximation, even if $\varepsilon = 0$ (which is, of course, the case for which such boundary conditions make sense). The emphasis in this paper is on the former case or at any rate on situations in which extra dissipation is needed; hence, (2.2) will be approximated by

$$L^h_{\beta h} U_{i,j} = -\beta h \Delta_h U_{i,j} + D_0^{x,h} U_{i,j} + D_0^{y,h} U_{i,j} = F_{i,j} \tag{2.3}$$

where $\beta = 0(1)$. Although (2.3) with Dirichlet boundary conditions on all of $\partial\Omega$, is formally (spatially) stable for $\beta > 0$, in practice there are ocsillations which are numerically unacceptable for β less than about 1/8 [Hy].

There are several choices for I^k_{k-1} and I^{k-1}_k which will reduce to the prescription in Sec. I, and we investigate several of them in Sec. III. Are there any that will work for (2.3)? There are two guiding principles in answering this question. The first is that if the equation happens to be positive definite (the "$\beta = \infty$" case of (2.3)), then the choices of L^k, I^k_{k-1}, and I^{k-1}_k should reduce to the choices made in Sec. I. The second principle is in the paper [B1]. There it was suggested that since the operator $L^h_{\beta h}$ has an upstream bias, the residual weighting operators I^{k-1}_k should have a downstream bias. Also it was suggested that a good choice for I^k_{k-1} should be bilinear interpolation. The goodness of these

choices was borne out by numerical experiments, showing them to be the best among many other alternatives. We emphasize, however, that the operators L^k were formed directly in [B1]; formation of L^{k-1} as $I_k^{k-1} L^k I_{k-1}^k$ was not investigated therein.

Let us refer to the grid with mesh spacing h in (2.3) as G^M (thus $h_M = h$) and write (2.3) as the matrix equation

$$L^M U^M = F^M \quad .$$

Consider the symmetric part of L^k, $\text{symm}(L^k) = \frac{1}{2}(L^k + (L^k)^*)$, $k \leq M$ and define I_{k-1}^k using $\text{symm}(L^k)$ instead of L^k. For example, (1.3) would now represent the template of $\text{symm}(L^k)$ at (IF+1,JF), etc. Note that in the case of (2.3), this prescription leads to bilinear interpolation for I_{k-1}^k. And, in the case of positive definite L^k, this prescription reduces to the previous one, since $\text{symm}(L^k) = L^k$ in that case.

For the definition of I_k^{k-1}, define J_{k-1}^k using $(L^k)^*$ instead of L^k. For example, (1.3) would now represent the template of $(L^k)^*$ at (IF+1,JF), etc. Then define $I_k^{k-1} = (J_{k-1}^k)^*$, and to complete the description, define as before $L^{k-1} = I_k^{k-1} L^k I_{k-1}^k$.

We will refer to the above derivations of I_{k-1}^k and J_{k-1}^k by saying that I_{k-1}^k is based on $\frac{1}{2}(L^k + (L^k)^*)$ and that J_{k-1}^k is based on $(L^k)^*$. Note that if (1.3) represents the template of $(L^k)^*$ at (IF+1), it can happen that C-N-S = 0; hence, our code checks for this case before dividing by C-N-S.

As in [D], the use of the right hand side in interpolation is important. Generally, the use of the right hand side provides an $O(h^2)$ correction to interpolation and is not worthwhile. In the black box approach, however, the right hand side next to the boundary can contain

boundary data, and in such cases, not using the right hand side can lead to O(1) interpolation errors. To use the right hand side, we use the same method as in [D]; if r^k is the residual at a given point and D^k is the diagonal coefficient of the difference operator, we add on a correction of r^k/D^k. For example, if (1.3) represents the template of symm(L^k) at (IF+1,JF), $r^k_{IF+1,JF}$ is the residual at (IF+1,JF), and $D^k_{IF+1,JF}$ is the diagonal coefficient at (IF+1,JF), then

$$(I^k_{k-1}V^{k-1})_{IF+1,JF} = \frac{(NW+W+SW)V^{k-1}_{IC,JC} + (NE+E+SE)V^{k-1}_{IC+1,JC}}{C-N-S} + r^k_{IF+1,JF}/D^k_{IF+1,JF} \quad . \tag{2.4}$$

There remains the issue of what kind of relaxation scheme to use. In general, point and line Gauss-Seidel, which were adequate for the positive definite case [D], will not suffice. Eq. (2.3) is of positive type if $\beta \geq \frac{1}{2}$, but there is a sweep direction dependence for Gauss-Seidel unless $\beta \geq 1$ [B1]. In any case the formation of L^{k-1} as $I^{k-1}_k L^k I^k_{k-1}$ may lead to Gauss-Seidel's being unacceptable on some coarser level even if it is acceptable on the finest level G^M. Hence, in general, it seems to be necessary to resort to a relaxation scheme like point or line Kaczmarz.

Point Kaczmarz [K] relaxation may be described as follows: let

$$\sum_{j=1}^{N} a_{ij}x_j = f_i \quad , \; i = 1,N$$

be the system of equations for which a solution is desired. Given x^k,

x^{k+1} is defined by finding successively, for each i, a δ_i such that

$$\sum_{j=1}^{N} a_{ij}(x_j^k + A_{ij}\delta_i) = f_i \tag{2.5}$$

is satisfied. Then the substitutions $x_j^{k+1} \leftarrow x^k + a_{ij}\delta_i$ are performed. It is straightforward to see that point Kaczmarz relaxation for $Ax = f$ is equivalent to point Gauss-Seidel for $AA^Ty = f$, where $A^Ty = x$; this relation gives a way of studying the smoothing rate for point Kacmarz.[B2]

Line Kaczmarz relaxation consists of satisfying (2.5) for a whole line of points at once. This requires the solution of a pentadiagonal system.

III. NUMERICAL EXPERIMENTS

To describe the results of the experiments, we introduce a few of the parameters used in the code; they have the same names as those used in [D].

TOL is the tolerance; iteration continues until the discrete L^2 norm of the residual on the finest grid is less than TOL or until too much work has been performed. Alternatively, if ALPHM > 0 is specified, the code will iterate until the discrete L^2 norm of $I_k^{k-1}F^k - L^{k-1}\tilde{I}_k^{k-1}U^k$ is less than ALPHM times $||r^M||$, the discrete L^2 norm of the residual on G^M, the finest grid; here $\tilde{I}_k^{k-1}$ denotes injection. The purpose of ALPHM is to detect when truncation error has been achieved, and the theoretical value for achieving truncation error is ALPHM $\leq$ 2; this value assumes that the equations are in undivided form and that the G^k residuals are computed dynamically (and thus roughly twice the size of the static G^k residuals).

IRELAX denotes the kind of relaxation. For IRELAX = 1, 2, 3, 4, see [D]. IRELAX = 5, 6, 7, or 8 denotes point Kaczmarz, line Kaczmarz by lines

in x, line Kaczmarz by lines in y, and line Kaczmarz by lines in x followed by line Kaczmarz by lines in y, respectively.

IVW denotes the kind of multigrid cycle employed. IVW = 1 or 2 denotes V or W cycles respectively.[D] In all except one of the examples, the code begins on the coarsest level and bootstraps itself up to the finest level before continuing with V or W cycles.

In most of the experiments below, we report the number of multigrid cycles, NCYC, which were used. NCYC-1 is the number of time the finest grid is visited. All of the examples take IU = ID = 1 and IM = 2; see Sec. I.

The first example is

$$-\beta h\Delta U^{\beta} + \frac{\partial U^{\beta}}{\partial x} + \frac{\partial U^{\beta}}{\partial y} = \cos x \sin y + \sin x \cos y \equiv F \qquad (3.1)$$

$$\text{in } \Omega = (0,3) \times (0,2)$$

$$U^{\beta} = \sin x \sin y \text{ on } \partial\Omega \quad .$$

The solution of (3.1) with $\beta = 0$ is $U^0(x,y) = \sin x \sin y$. As described in Sec. II,

$$\frac{\partial U^{\beta}}{\partial x} \text{ and } \frac{\partial U^{\beta}}{\partial y}$$

are differenced with central differencing and ΔU is replaced by the five point Laplacian. As in [D] the Dirichlet data is handled by absorbing it into F; thus, the boundary is not treated as part of the grid. The results of using the method described in Sec. II are summarized in Table 3.1. To facilitate comparison with the results in [B1] we give the $L^1(\Omega')$ and

TABLE 3.1
Results for method applied to (3.1)

Problem size and parameters	$L^1(\Omega')$ error	max(Ω') error	CPU time in seconds on CRAY-1	CF in first and last cycle and number of cyles
47x31 (M=4), TOL = 10^{-6} IRELAX=5, IVW=1, β=1	5.4,-2	8.8,-2	.20	.17, .30, NCYC=11
47x31 (M=4), TOL = 10^{-6} IRELAX=5, IVW=1, β=½	2.6,-2	4.4,-2	.22	.20, .34, NCYC=13
47x31 (M=4), TOL = 10^{-6} IRELAX=5, IVW=1, β=¼	1.2,-2	2.2,-2	.33	.29, .49, NCYC=20
47x31 (M=4), TOL = 10^{-6} IRELAX=5, IVW=1, β=1/8	5.8,-3	1.0,-2	.33*	.42, .64, NCYC=20
47x31 (M=4), TOL = 10^{-6} IRELAX=5, IVW=2, β=1/16	---	---	---	4.0, 4.5, NCYC=20
47x31 (M=4), TOL = 10^{-6} IRELAX=5, IVW=2, β = 1	5.4,-2	8.8,-2	.27	.15, .31, NCYC=11
47x31 (M=4), TOL = 10^{-6} IRELAX=5, IVW=2, β=½	2.6,-2	4.4,-2	.26	.18, .33, NCYC=11
47x31 (M=4), TOL = 10^{-6} IRELAX=5, IVW=2, β=¼	1.2,-2	2.2,-2	.43	.27, .50, NCYC=19
47x31 (M=4), TOL = 10^{-6} IRELAX=5, IVW=2, β=1/8	5.8,-3	1.0,-2	.45*	.41, .65, NCYC=20
23x15 (M=3), TOL = 10^{-6} IRELAX=5, IVW=1, β=½	5.0,-2	8.1,-2	.06	.20, .29, NCYC=9
11x7 (M=2), TOL = 10^{-6} IRELAX=5, IVW=1, β=½	1.2,-1	1.6,-1	.02	.20, .23, NCYC=13
47x31 (M=4), ALPHM=2.0 IRELAX=5, IVW=1, β=1	5.6,-2	9.0,-2	.06	.17, .19, NCYC=2
47x31 (M=4), ALPHM=2.0 IRELAX=5, IVW=1, β=½	3.2,-2	4.5,-2	.06	.20, 22, NCYC=2
47x31 (M=4), ALPHM=2.0 IRELAX=5, IVW=1, β=¼	2.4,-2	2.3,-2	.06	.29, .34, NCYC=2

Table 3.1 (concluded)

Problem size and parameters	$L^1(\Omega')$ error	$\max(\Omega')$ error	CPU time in seconds on CRAY-1	CF in first and last cycle and number of cyles
47x31 (M=4), ALPHM=2.0 IRELAX=5, IVW=1, β=1/8	4.7,-2	6.4,-2	.05	.42, .42, NCYC=1
47x31 (M=4), ALPHM=2.0 IRELAX=5, IVW=2, β=1	5.3,-2	8.8,-2	.08	.15, .14, NCYC=2
47x31 (M=4), ALPHM=2.0 IRELAX=5, IVW=2, β=½	2.5,-2	4.4,-2	.08	.26, .18, NCYC=2
47x31 (M=4), ALPHM=2.0 IRELAX=5, IVW=2, β=⅛	1.2,-2	2.2,-2	.08	.27, .27, NCYC=2
47x31 (M=4), ALPHM=2.0 IRELAX=5, IVW=3, β=1/8	2.7,-2	3.6,-2	.06	.41, .41, NCYC=1

* Failed to reduce norm of residual below 10^{-6} in alloted number of cycles.

$\max(\Omega')$ norms of the error $u^h - U^0$ in the subdomain $\Omega' = \{(x,y): 0 \leq x \leq 2, 0 \leq y \leq 4/3\}$, where the $\max(\Omega')$ error is $\max_{(ih,jh)\in \Omega'} |u^h_{i,j} - U^0_{i,j}|$. The CPU times in seconds on the CRAY-1 computer are given mostly for purposes of comparison between different runs, since little effort has been invested in writing efficient code; the times are scalar times since no vectorization has been introduced.

In the last column are given CF in the first and last cycle and the number of cycles; CF is the convergence factor, the reduction in the discrete L^2 norm of the residual in a single cycle. Twenty was the maximum number of cycles allowed. The notation 5.4,-2, for example, is used to indicate 5.4×10^{-2}.

Note that the behavior of the error with the number of unknowns and β is as theory predicts, and the CPU times support the linear dependence of the reduction of the error on the number of unknowns. For TOL = 10^{-6} there

is no advantage to W-cycles over V-cycles, but for ALPHM =2, the W-cycles give better results for $\beta = \frac{1}{4}$ but still fail to achieve truncation error as advertised when $\beta = 1/8$.

Note also that the convergence factor per cycle is not nearly as good as in the positive definite case; in that case for smooth coefficients and V-cycles, a convergence factor of around .1 was observed. It is worth repeating the argument in [B1] that suggests that, in fact, one cannot expect such good convergence factors in the singular perturbation case. Suppose there are two levels and that (2.3) is the discretization on the finest level. The argument in [B1] uses first-differential approximation (FDA) analysis in which the difference operators are replaced by their first-differential approximations, inter-grid transfer operators are ignored, and relaxation is ignored. The FDA approximation to $L_h^{\beta h}$ is $L^h = -\beta h\Delta + \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$. Consider a characteristic smooth error component V^h, i.e.,

$$\frac{\partial V^h}{\partial x} + \frac{\partial V^h}{\partial y} = 0 \quad ;$$

its residual is $-\beta h\Delta V^h$. The coarse grid equation, assuming the same discretization is

$$L^{2h}V^{2h} \equiv -\beta(2h)\Delta V^{2h} + \frac{\partial V^{2h}}{\partial x} + \frac{\partial V^{2h}}{\partial y} = -\beta h\Delta V \quad ,$$

and its solution is $V^{2h} = \frac{1}{2}V^h$. Thus the new error on the fine grid after a full multigrid cycle is

$$V^h_{NEW} = V^h - V^{2h} = \tfrac{1}{2}V^h \quad .$$

The extension of this argument to $M > 2$ levels suggests that if V cycles are used, convergence no better than $1-M^{-1}$ could be expected per cycle; [B1] concludes, however, that this analysis is too pessimistic and also points to numerical experiments in which using W cycles instead of V cycles gave rise to convergence rates better than .5 per cycle.

In order to apply the above analysis to the present situation, one must discover how much dissipation is generated numerically. We have done this numerically for the 47×31 ($M=4$) case of Table 3.1, and the results (away from boundaries) are presented in Table 3.2; the notation is:

$$\Delta_h^{sk} U_{i,j} = \frac{1}{2h^2}(U_{i-1,j-1} + U_{i-1,j+1} + U_{i+1,j-1} + U_{i+1,j+1} - 4U_{i,j}) \quad ,$$

$$D_h^{xy} = 2D_h^{0,x} D_h^{0,y} \quad ,$$

$$D_h^{sw} = \theta\Delta_h + (1-\theta)\Delta_h^{sk} + D_h^{xy} \quad , \text{ for some } \theta \quad , \; 0 \le \theta \le 1 \quad ,$$

$$\Delta_h^{lc} = \theta\Delta_h + (1-\theta)\Delta_h^{sk} \quad , \text{ for some } \theta \quad , \; 0 \le \theta \le 1 \quad ;$$

note that $-D_h^{sw}$ is dissipative in the streamwise direction and an approximation to $-(\frac{\partial}{\partial x} + \frac{\partial}{\partial y})(\frac{\partial}{\partial x} + \frac{\partial}{\partial y})$; see [B1] and [H].

Thus it appears that the method tends to a "β" of .5, although the form of the dissipation is more complicated than in (2.3). Also of interest is the behavior of the dissipation with respect to the original β's being greater than or less than $\frac{1}{2}$. In the former case the β's decrease monotonically to $\frac{1}{2}$; in the latter case there is an initial increase and then a decrease to $\frac{1}{2}$; if $\beta = \frac{1}{2}$ on the finest grid, it remains $\frac{1}{2}$ on all grids.

TABLE 3.2$_k$
Form of dissipation on G^k for various β's

β	k	Dissipation	Strength of dissipation
1	3	$-h(.3125\Delta_h + .3125\Delta_h^{sk} + .625D_h^{xy}) = -h(.625D_h^{sw})$	.625h
1	2	$-h(.1666\Delta_h + .3460\Delta_h^{sk} + .2625D_h^{xy}) = -h(.2625D_h^{sw} + .25\Delta_h^{lc})$	.5125h
1	1	$-h(.1016\Delta_h + .3985\Delta_h^{sk} + .3752D_h^{xy}) = -h(.375D_h^{sw} + .125\Delta_h^{lc})$	.5000h
½	3	$-h(.25\Delta_h + .25\Delta_h^{sk} + .25D_h^{xy}) = -h(.25D_h^{sw} + .25\Delta_h^{lc})$	.5h
½	2	$-h(.125\Delta_h + .375\Delta_h^{sk} + .375D_h^{xy}) = -h(.375D_h^{sw} + .125\Delta_h^{lc})$	.5h
½	1	$-h(.0625\Delta_h + .4375\Delta_h^{sk} + .4375D_h^{xy}) = -h(.4375D_h^{sw} + .0625\Delta_h^{lc})$	.5h
¼	3	$-h(.3125\Delta_h + .3125\Delta_h^{sk} + .5D_h^{xy}) = -h(.5D_h^{sw} + .125\Delta_h^{lc})$	.625h
¼	2	$-h(.0897\Delta_h + .4228\Delta_h^{sk} + .45D_h^{xy}) = -h(.45D_h^{sw} + .0625\Delta_h^{lc})$	.5125h
¼	1	$-h(.0371\Delta_h + .4630\Delta_h^{sk} + .4689D_h^{xy}) = -h(.4689D_h^{sw} + .0312\Delta_h^{lc})$	.5002h
1/8	3	$-h(.5313\Delta_h + .5313\Delta_h^{sk} + D_h^{xy}) = -h(D_h^{sw} + .0625\Delta_h^{lc})$	1.0625h
1/8	2	$-h(.0907\Delta_h + .5582\Delta_h^{sk} + .6177D_h^{xy}) = -h(.6177D_h^{sw} + .0313\Delta_h^{lc})$	.6490h
1/8	1	$-h(.0243\Delta_h + .4928\Delta_h^{sk} + .5015D_h^{xy}) = -h(.5015D_h^{sw} + .0156\Delta_h^{lc})$	.5170h
1/16	3	$-h(.1388\Delta_h + .9384\Delta_h^{sk} + 1.0616D_h^{xy}) = -h(1.0616D_h^{sw} + .0156\Delta_h^{lc})$	1.0772h

The approximation that the method yields to the convective terms on coarser grids is also interesting. It yields approximations to $U_x + U_y$ which are a linear combination of the usual central differencing and the difference $U_{i+1,j+1} - U_{i-1,j-1}$.

One other question is why the data in Table 3.1 does not agree with the experimental conclusion in [B1] that the use of W cycles leads to a convergence factor of better than .5 per cycle. It is only for $\beta = 1/8$ that the convergence factor is worse than .5. Thus, one explanation that comes to mind is as follows. The correction term in the interpolation

(2.4) is based on point Gauss-Seidel. For small β, perhaps a correction term based on point Kaczmarz is more appropriate since it would magnify the offending frequencies less; in (2.4) the difference would be that instead of division by $D^k_{IF+1,JF}$, one would multiphy by $D^k_{IF+1,JF}$ divided by the sum of the squares of the coefficients of L^k at (IF+1,JF). The results of this change are summarized in Table 3.3. From the table, we see improvement for W cycles in comparison with V cycles - not the case with Table 3.1, but using V cycles still uses less CPU time than using W cycles for TOL = 10^{-6}. Again for ALPHM = 2, use of W cycles or V cycles does not give truncation error as advertised for $\beta = 1/8$. Note also that CF is larger for V cycles in Table 3.3 than in Table 3.1. From this example and others we have investigated, the previous method seems better on the average than this variant; hence, it is the one that is implemented in the current version of the code.

TABLE 3.3
Results for method applied to (3.1) with modified use of right hand side in interpolation

Problem size and parameters	$L^1(\Omega')$ error	max(Ω') error	CPU time in seconds on CRAY-1	CF in first and last cycle and number of cyles
47x31 (M=4), TOL = 10^{-6}, IRELAX=5, IVW=1, β=1	5.4,-2	8.8,-2	.21	.16, .37, NCYC=12
47x31 (M=4), TOL = 10^{-6}, IRELAX=5, IVW=1, β=½	2.6,-2	4.4,-2	.21	.18, .39, NCYC=12
47x31 (M-4), TOL = 10^{-6}, IRELAX=5, IVW=1, β=¼	1.2,-2	2.2,-2	.25	.23, .50, NCYC=15
47x31 (M=4), TOL = 10^{-6}, IRELAX=5, IVW=1, β=1/8	5.8,-3	1.0,-2	.33*	.29, .67, NCYC=20

Table 3.3 concluded

Problem size and parameters	$L^1(\Omega')$ error	$\max(\Omega')$ error	CPU time in seconds on CRAY-1	CF in first and last cycle and number of cyles
47x31 (M=4), TOL = 10^{-6}, IRELAX=5, IVW=1, β=1/16	2.5,-3	8.8,-3	.33*	.53, .98, NCYC=20
47x31 (M=4), TOL = 10^{-6}, IRELAX=5, IVW=1, β=1/32	---	---	---	12.0, 10.0, NCYC=20
47x31 (M=4), TOL = 10^{-6}, IRELAX=5, IVW=2, β=1	5.4,-2	8.8,-2	.29	.15, .37, NCYC=13
47x31 (M=4), TOL = 10^{-6}, IRELAX=5, IVW=2, β=½	2.6,-2	4.4,-2	.28	.28, .39, NCYC=15
47x31 (M=4), TOL = 10^{-6}, IRELAX=5, IVW=2, β=¼	1.2,-2	2.2,-2	.35	.21, .40, NCYC=16
47x31 (M=4), TOL = 10^{-6}, IRELAX=5, IVW=2, β=1/8	5.8,-3	1.0,-2	.39	.29, .49, NCYC=17
47x31 (M=4), TOL = 10^{-6}, IRELAX=5, IVW=2, β=1/16	2.5,-3	8.8,-3	.45*	.61, .83, NCYC=20
47x31 (M=4), ALPHM = 2., IRELAX=5, IVW=1, β=1	5.5,-2	8.8,-2	.06	.16, .22, NCYC=3
47x31 (M=4), ALPHM = 2., IRELAX=5, IVW=1, β=½	2.8,-2	4.2,-2	.06	.18, .19, NCYC=3
47x31 (M=4), ALPHM = 2., IRELAX=5, IVW=1, β=¼	1.1,-2	1.6,-2	.05	.23, .23, NCYC=2
47x31 (M=4), ALPHM = 2., IRELAX=5, IVW=1, β=1/8	4.2,-2	5.4,-2	.05	.29, .29, NCYC=2
47x31 (M=4), ALPHM = 2., IRELAX=5, IVW=2, β=1	5.3,-2	8.8,-2	.08	.07, .16, NCYC=3
47x31 (M=4), ALPHM = 2., IRELAX=5, IVW=2, β=½	2.6,-2	4.4,-2	.08	.15, .11, NCYC=3
47x31 (M=4), ALPHM = 2., IRELAX=5, IVW=2, β=¼	1.2,-2	2.3,-2	.06	.21, .21, NCYC=2
47x31 (M=4), ALPHM = 2., IRELAX=5, IVW=2, β=1/8	2.9,-2	3.5,-2	.06	.29, .29, NCYC=2

* Failed to reduce norm of residual below 10^{-6} in allotted number of cycles.

Another final matter we wish to investigate is

$$-\beta h\Delta U^{\beta} + \frac{\partial U^{\beta}}{\partial x} + \frac{\partial U^{\beta}}{\partial y} = \cos x \sin y + \sin x \cos y \text{ in } \Omega = (0,3) \times (0,2)$$

$$U^{\beta} = \sin x \sin y \text{ on } y = 0 \text{ and on } x = 0 \quad .$$

We compare two treatments of the boundaries $x = 3$ and $y = 2$, the Dirichlet case, where U^{β} is specified as before and the outflow boundary condition case, where the equation on $x = 3$ and $y = 2$ is taken to be

$$\frac{1}{h}(U^{\beta}_{i,j} - U^{\beta}_{i,j-1}) + \frac{1}{h}(U^{\beta}_{i,j} - U^{\beta}_{i-1,j}) = \cos x_i \sin y_j + \sin x_i \cos y_j \quad .$$

The results are summarized in Table 3.4. The error is measured over all of Ω, and as expected the Dirichlet boundary condition case is bad for $\beta < \frac{1}{2}$ while the outflow case continues to give good results for such β's.

TABLE 3.4
Comparison of Dirichlet and outflow boundary conditions

Problem size and parameters	$L^1(\Omega')$ error	$\max(\Omega')$ error	CPU time in seconds on CRAY-1	CF in first and last cycle and number of cyles
47x31 (M=4), TOL = 10^{-6}, IRELAX=5, IVW=1, β=1., Dirichlet B.C.	2.3,-1	1.8,-1	.20	.17, .30, NCYC=11
47x31 (M=4), TOL = 10^{-6}, IRELAX=5, IVW=1, β=½ Dirichlet B.C.	1.1,-1	7.4,-2	.22	.20, .34, NCYC=13
47x31 (M=4), TOL = 10^{-6}, IRELAX=5, IVW=1, β=¼ Dirichlet B.C.	8,0,-2	3.1,-1	.33	.29, .49, NCYC=13

Table 3.4 concluded

Problem size and parameters	$L^1(\Omega')$ error	$\max(\Omega')$ error	CPU time in seconds on CRAY-1	CF in first and last cycle and number of cyles
47x31 (M=4), TOL = 10^{-6}, IRELAX=5, IVW=1, β=1/8 Dirichlet B.C.	1.1,-1	5.4,-1	.42*	.33, .64, NCYC=20
47x31 (M=4), TOL = 10^{-6}, IRELAX=5, IVW=1, β=1 outflow B.C.	2.3,-1	1.4,-1	.21	.30, .29, NCYC=12
47x31 (M=4), TOL = 10^{-6}, IRELAX=5, IVW=1, β=1/2 outflow B.C.	1.1,-1	7.4,-2	.22	.33, .34, NCYC=13
47x31 (M=4), TOL = 10^{-6}, IRELAX=5, IVW=2, β=¼ outflow B.C.	5.5,-2	3.8,-2	.33	.39, .49, NCYC=19
47x31 (M=4), TOL = 10^{-6}, IRELAX=5, IVW=2, β=1/8 outflow B.C.	2.7,-2	3.1,-2	.33*	.42, .71, NCYC=20

* Failed to reduce norm of residual below 10^{-6} in alloted number of cycles.

Now let us mention various alternatives to the method of Sec. II, which bases I_{k-1}^k on $\frac{1}{2}(L^k + (L^k)^*)$ and uses $I_k^{k-1} = (J_{k-1}^k)^*$ where J_{k-1}^k is based on $(L_k)^*$. These methods all reduce to the method of Sec. I in the positive definite case, and they all take $I_k^{k-1} = (J_{k-1}^k)^*$ and $L^k = I_k^{k-1}$. The first bases $I_{k-1}^k = J_{k-1}^k$ on L^k; this method has a CF of .71 per V cycle for $\beta = 1$ and a CF of 1.1 for $\beta = \frac{1}{2}$. The second bases I_{k-1}^k on L^k and J_{k-1}^k on $(L^k)^*$; it has a CF of 4.8 per V cycle for $\beta = 1$. The third bases both $I_{k-1}^k = J_{k-1}^k$ on $\frac{1}{2}(L^k + (L^k)^*)$; for $\beta = 1$, $\frac{1}{2}$, $\frac{1}{4}$, and 1/8, it yields CF's of .36, .56, .89, and 1.7 per V cycle, respectively. This last method, since it is related to blind application of the finite element multigrid method with piecewise bilinear elements deserves at least a little comment. If we

compute numerically the dissipation of the 47x31 (M=4) case, away from boundaries, with $\beta = 1$, as done in Table 3.2, the result for $k = 3$ is $-h(\frac{1}{2}\Delta_h + \frac{1}{2}\Delta_h^{sk})$ which has strength $\frac{1}{2}h$; the result for $k = 2$ is $-h(.09375\Delta_h + .15625\Delta_h^{sk})$ which has strength $\frac{1}{8}h$; thus the "β" is apparently halved on each level, giving rise to problems that are harder and harder to solve.

The next example is

$$\begin{aligned} &-\beta h\Delta U^\beta + \frac{\partial U^\beta}{\partial x} + \frac{\partial U^\beta}{\partial y} = 0 \text{ in } \Omega = (0,3) \times (0,2) \\ &U^\beta = H(x-y-1) \text{ on } \partial\Omega \quad , \end{aligned} \tag{3.2}$$

where $H(\xi) = 0$ if $\xi \leq 0$ and $H(\xi) = 1$ if $\xi > 0$. The solution for $\beta = 0$ is $H(x-y-1)$. The results are summarized in Table 3.5. Section 4.3 of [B1] suggests that special care may have to be taken with interpolation near discontinuities, as in (3.2). The reason for this suggestion is that [B1] uses the same discretization on each grid. Since the formation of L^{k-1} as $I_k^{k-1} L^k I_{k-1}^k$ tends to smear out discontinuities, such special care with interpolation is apparently not necessary for the method of Sec. II. This is another advantage of the black box approach.

TABLE 3.5
Results of the method applied to (3.2)

Problem size and parameters	$L^1(\Omega')$ error	$\max(\Omega')$ error	CPU time in seconds on CRAY-1	CF in first and last cycle and number of cyles
47x31 (M=4), TOL = 10^{-6}, IRELAX=5, IVW=1, β=1.,	2.7,-1	5.4,-1	.22	.21, .31, NCYC=13
47x31 (M=4), TOL = 10^{-6}, IRELAX=5, IVW=1, β=½	2.0,-1	5.0,-1	.22	.16, .34, NCYC=13

Table 3.5 concluded

Problem size and parameters	$L^1(\Omega')$ error	$\max(\Omega')$ error	CPU time in seconds on CRAY-1	CF in first and last cycle and number of cyles
47x31 (M=4), TOL = 10^{-6}, IRELAX=5, IVW=1, $\beta=\frac{1}{4}$	1.5,-1	4.7,-1	.33	.20, .48, NCYC=19
47x31 (M=4), TOL = 10^{-6}, IRELAX=5, IVW=1, β=1/8	1.1,-1	4.5,-1	.33*	.34, .52, NCYC=20
47x31 (M=4), TOL = 10^{-6}, IRELAX=5, IVW=1, β=1/16	---	---	---	3.8, 4.1, NCYC=20
47x31 (M=4), ALPHM = 2., IRELAX=5, IVW=1, β=1	2.7,-1	5.6,-1	.05	.21, .21, NCYC=1
47x31 (M=4), ALPHM = 2., IRELAX=5, IVW=2, $\beta=\frac{1}{2}$	2.0,-1	5.1,-1	.05	.16, .16, NCYC=1
47x31 (M=4), ALPHM = 2., IRELAX=5, IVW=2, $\beta=\frac{1}{4}$	1.6,-1	4.9,-1	.05	.20, .20, NCYC=1
47x31 (M=4), ALPHM = 2., IRELAX=5, IVW=2, β=1/8	1.4,-1	4.8,-1	.05	.34, .34, NCYC=1

* Failed to reduce norm of residual below 10^{-6} in alloted number of cycles.

The next example is

$$-\beta h\frac{\partial}{\partial x}(|y|\frac{\partial U^\beta}{\partial x}) -\beta h\frac{\partial}{\partial y}(|x|\frac{\partial U^\beta}{\partial y}) - y\frac{\partial U^\beta}{\partial x} + x\frac{\partial U^\beta}{\partial y}$$

$$= -y\cos x \sin y + x \cos y \sin x \text{ in } \Omega = (-\tfrac{1}{2},\tfrac{1}{2}) \times (-\tfrac{1}{2},\tfrac{1}{2})$$

$$U^\beta = \sin x \sin y \text{ on } \partial\Omega \quad . \tag{3.3}$$

The solution of (3.3) with $\beta = 0$ is $U^0(x,y) = \sin x \sin y$. Central differencing is used everywhere for (3.3). Note that the characteristics in (3.3) are circles. The form of dissipation was recommended by Hyman [Hy]. The results are summarized in Table 3.6. Note that alternating line Kaczmarz is necessary in this problem since $|x| \ll |y|$ in parts of the domain and $|x| \gg |y|$ in other parts of the domain. Note also that the ALPHM = 2. case here is an utter failure.

TABLE 3.6
Results of the method applied to (3.3)

Problem size and parameters	$L^1(\Omega')$ error	$\max(\Omega')$ error	CPU time in seconds on CRAY-1	CF in first and last cycle and number of cyles
47x47 (M=5), TOL = 10^{-6} IRELAX=5, IVW=1, β=1	3.8,-4	1.6,-3	.50*	.24, .88, NCYC=20
47x47 (M=5), TOL = 10^{-6} IRELAX=6, IVW=1, β=1	3.8,-4	1.6,-3	1.4	.23, .55, NCYC=19
47x47 (M=5), TOL = 10^{-6} IRELAX=8, IVW=1, β=1	3.8,-4	1.6,-3	1.6	.14, .34, NCYC=10
47x47 (M=5), TOL = 10^{-6} IRELAX=8, IVW=1, β=½	2.0,-4	1.0,-3	2.2	.27, .54, NCYC=14
47x47 (M=5), TOL = 10^{-6} IRELAX=8, IVW=2, β=¼	---	---	3.0*	.28, .86, NCYC=20
47x47 (M=5), TOL = 10^{-6} IRELAX=8, IVW=2, β=1/8	---	---	---	5.0, 5.6, NCYC=20
47x47 (M=5), TOL = 10^{-6} IRELAX=8, IVW=2, β=1	3.8,-4	1.6,-3	1.4	.13, .11, NCYC=6
47x47 (M=5), TOL = 10^{-6} IRELAX=8, IVW=2, β=½	2.0,-4	1.0,-3	1.9	.20, .25, NCYC=9
47x47 (M=5), TOL = 10^{-6} IRELAX=8, IVW=2, β=¼	1.0,-4	7.4,-4	3.1	.22, .53, NCYC=15
47x47 (M=5), ALPHM = 2.0 IRELAX=8, IVW=1, β=1	2.0,-3	1.1,-2	.37	.14, .14, NCYC=1
47x47 (M=5), ALPHM = 2.0 IRELAX=8, IVW=1, β=½	4.5,-3	3.5,-2	.37	.27, .27, NCYC=1
47x47 (M=5), ALPHM=2.0 IRELAX=8, IVW=1, β=1	1.3,-3	9.1,-3	.43	.13, .13, NCYC=1
47x47 (M=5), ALPHM=2.0 IRELAX=8, IVW=1, β=½	2.9,-3	2.4,-2	.43	.20, .20, NCYC=1
47x47 (M=5), ALPHM=2.0 IRELAX=8, IVW=1, β=¼	9.6,-3	3.3,-2	.43	.22, .22, NCYC=1

* Failed to reduce norm of residual below 10^{-6} in alloted number of cycles.

The next example is

$$-\beta h\Delta U^{\beta} + \frac{\partial U^{\beta}}{\partial x} + \frac{\partial U^{\beta}}{\partial y} = \cos x \sin y + \cos y \sin x \equiv F(x,y)$$

$$\text{in } \Omega = \{(x,y):\sqrt{x^2+y^2} \leq \tfrac{1}{2}\}$$

$$U^{\beta} = \sin x \sin y \equiv g(x,y) \text{ on } \partial\Omega \quad . \tag{3.4}$$

The solution of (3.4) with $\beta = 0$ is $U^0(x,y) = \sin x \sin y$. We embed Ω in $\tilde{\Omega} = (-.5,.5) \times (-.5,.5))$ as in the similar example in [D]. At points in $\tilde{\Omega} \setminus \bar{\Omega}$ we write down the equation $u = 0$; at points in Ω whose north, south, east, and west neighbors are in $\bar{\Omega}$, we use the usual central differencing. For simplicity we use the simplest treatment for points that do not fall into either of the above sets. Consider, for example, a point $U_{i,j}$ in Ω whose neighbor $U_{i+1,j}$ is not in $\bar{\Omega}$ and let the distance from $U_{i,j}$ to $\partial\Omega$ be θh. Use of the relation $g((i+\theta)h,jh)) \simeq (1-\theta)U_{i,j} + \theta U_{i+1,j}$ gives the following difference equation at (ih,jh):

$$h(-\beta-\tfrac{1}{2})U_{i-1,j} + (\beta h(3+\frac{1}{\theta}) + \frac{h}{2}(1-\theta))U_{i,j} + h(-\beta+\tfrac{1}{2})U_{i,j+1} + h(-\beta-\tfrac{1}{2})U_{i,j-1}$$

$$= h^2F(ih,jh) + \frac{1}{\theta}h(\beta-\tfrac{1}{2})g((i+\sigma)h,jh) \quad .$$

The results are summarized in Table 3.7, where $\Omega' = \{(x,y):\sqrt{x^2+y^2} \leq \tfrac{1}{4}\}$.

The next example is the same except that in this case we use mapping to solve it. That is, we map the boundary of Ω onto the boundary of $\Omega'' = (0,1) \times (0,1)$ giving x and y as a function of ξ and η on $\partial\Omega''$ and

TABLE 3.7
Results of the method applied to (3.4)

Problem size and parameters	$L^1(\Omega')$ error	$\max(\Omega')$ error	CPU time in seconds on CRAY-1	CF in first and last cycle and number of cyles
47x47 (M=5), TOL = 10^{-6} IRELAX=5, IVW=2, $\beta=\frac{1}{2}$	2.1,-4	7.6,-4	.41	.24, .30, NCYC=11
23x23 (M=4), TOL = 10^{-6} IRELAX=5, IVW=2, $\beta=\frac{1}{2}$	3.1,-4	1.0,-3	.12	.18, .30, NCYC=11
11x11 (M=3), TOL = 10^{-6} IRELAX=5, IVW=2, $\beta=\frac{1}{2}$	4.0,-4	1.2,-3	.03	.16, .26, NCYC=9
47x47 (M=5), ALPHM = 2. IRELAX=5, IVW=2, $\beta=\frac{1}{2}$	6.2,-4	1.2,-2	.03	.24, .24, NCYC=1
23x23 (M=4), ALPHM = 2. IRELAX=5, IVW=2, $\beta=\frac{1}{2}$	7.3,-4	4.8,-3	.03	.18, .18, NCYC=1
11x11 (M=3), ALPHM = 2. IRELAX=5, IVW=2, $\beta=\frac{1}{2}$	8.4,-4	2.6,-3	.01	.16, .16, NCYC=1

solving approximately the problem

$$\begin{aligned} \Delta x &= 0 \quad , (\xi,\eta) \in \Omega'' \\ x &= x(\xi,\eta) \quad , (\xi,\eta) \in \partial\Omega'' \\ \\ \Delta y &= 0 \quad , (\xi,\eta) \in \Omega'' \\ y &= y(\xi,\eta) \quad , (\xi,\eta) \in \partial\Omega'' \quad . \end{aligned} \tag{3.5}$$

We do this by discretizing Ω'', approximating (3.5) by five point Laplacians and specifying

$$x(ih,1) = \tfrac{1}{2}\cos(\frac{3\pi}{4} - ih\frac{\pi}{2}),\ y(ih,1) = \tfrac{1}{2}\sin(\frac{3\pi}{4} - ih\frac{\pi}{2})$$

$$x(ih,0) = \tfrac{1}{2}\cos(\frac{5\pi}{4} + ih\frac{\pi}{2}),\ y(ih,0) = \tfrac{1}{2}\sin(\frac{5\pi}{4} + ih\frac{\pi}{2})$$

$$x(0,ih) = \tfrac{1}{2}\cos(\frac{5\pi}{4} - ih\frac{\pi}{2}),\ y(0,ih) = \tfrac{1}{2}\sin(\frac{5\pi}{4} - ih\frac{\pi}{2})$$

$$x(1,ih) = \tfrac{1}{2}\cos(\frac{-\pi}{4} + ih\frac{\pi}{2}),\ y(1,ih) = \tfrac{1}{2}\sin(\frac{-\pi}{4} + ih\frac{\pi}{2}).$$

The transformed equation in the ξ-η coordinate system is:

$$-\frac{\partial}{\partial\xi}(|y_\eta - x_\eta|\frac{\partial U^\beta}{\partial\xi}) - \frac{\partial}{\partial\eta}(|x_\xi - y_\xi|\frac{\partial U^\beta}{\partial\eta}) + (y_\eta - x_\eta)\frac{\partial U^\beta}{\partial\xi} + (x_\xi - y_\xi)\frac{\partial U^\beta}{\partial\eta} = FJ,\ (\xi,\eta)\varepsilon\ \Omega''$$

$$U^\beta(\xi,\eta) = g(\xi,\eta),\ (\xi,\eta)\ \varepsilon\ \partial\Omega'' \qquad (3.6)$$

where $J = x_\xi y - x_\eta y_\xi$; note that we have chosen to compute the dissipation in the transformed system rather than transform the dissipation from (3.4). Eq. (3.6) is differenced in cell-centered form. The results are summarized in Table 3.8. Note that because of the singularity introduced by the mapping, the max(Ω) error does not decrease with decreasing h or β.

TABLE 3.8
Results of method applied to (3.4) using mapping

Problem size and parameters	$L^1(\Omega')$ error	max(Ω') error	CPU time in seconds on CRAY-1	CF in first and last cycle and number of cyles
48x48 (M=5), TOL = 10^{-6} IRELAX=5, IVW=2, β=1	8.9,-4	1.3,-2	.65	.28, .41, NCYC=13
48x48 (M=5), TOL = 10^{6} IRELAX=5, IVW=2, β=½	5.1,-4	2.0,-2	.80	.35, .50, NCYC=19
48x48 (M=5), TOL = 10^{-6} IRELAX=5, IVW=2, β=¼	3.2,-4	2.8,-2	.81*	.43, .71, NCYC=20
48x48 (M=5), TOL = 10^{-6} IRELAX=5, IVW=2, β=1/8	---	---	---	1.01, 1.8, NCYC=20

Table 3.8 concluded

Problem size and parameters	$L^1(\Omega')$ error	$\max(\Omega')$ error	CPU time in seconds on CRAY-1	CF in first and last cycle and number of cyles
48x48 (M=5), TOL = 10^{-6} IRELAX=5, IVW=2, β=1	8.9,-4	1.3,-2	.25	.28, .41, NCYC=5
48x48 (M=5), TOL = 10^{-6} IRELAX=5, IVW=2, β=½	5.1,-4	2.0,-2	.32	.35, .50, NCYC=7
48x48 (M=5), TOL = 10^{-6} IRELAX=5, IVW=2, β=¼	3.2,-4	2.8,-2	.29	.43, .71, NCYC=6
24x24 (M=4), TOL = 10^{-6} IRELAX=5, IVW=2, β=½	9.6,-4	1.9,-2	.19	.33, .45, NCYC=16
12x12 (M=3), TOL = 10^{-6} IRELAX=5, IVW=2, β=½	1.8,-3	1.9,-2	.05	.29, .38, NCYC=13

* Failed to reduce norm of residual below 10^{-6} in alloted number of cycles.

The next example uses the rather distorted Lagrangian mesh in Fig. 12 of [D]. We use the same differencing and F and g as for (3.6). Since we do not know the exact solution, we present only convergence data in Table 3.9.

TABLE 3.9
Method applied to Lagrangian mesh

Problem size and parameters	CPU time in seconds on CRAY-1	CF in first and last cycle and number of cycles
13x13 (M=3), TOL = 10^{-6} IRELAX=5, IVW=1, β=1	.05	.26, .41, NCYC=17
13x13 (M=3), TOL = 10^{-6} IRELAX=5, IVW=1, β=½	.06	.29, .43, NCYC=20
13x13 (M=3), TOL = 10^{-6} IRELAX=5, IVW=1, β=¼	.06*	.34, .79, NCYC=20
13x13 (M=3), TOL = 10^{-6} IRELAX=5, IVW=1, β=1/8	---	.96, 2.7, NCYC=20

Problem size and parameters	CPU time in seconds on CRAY-1	CF in first and last cycle and number of cycles
13x13 (M=3), TOL = 10^{-6} IRELAX=5, IVW=1, β=1	.06	.26, .42, NCYC=17
13x13 (M=3), TOL = 10^{-6} IRELAX=5, IVW=1, β=½	.07	.29, .44, NCYC=20
13x13 (M=3), TOL = 10^{-6} IRELAX=5, IVW=1, β=¼	.07*	.33, .68, NCYC=20

* Failed to reduce norm of residual below 10^{-6} in alloted number of cycles.

The final example is a scalar version of the two-dimensional Burger's equation:

$$-\beta h\nabla\cdot(\tfrac{1}{2}|U^{\beta}|\nabla U^{\beta}) + \tfrac{1}{2}\frac{\partial(U^{\beta})^2}{\partial x} + \tfrac{1}{2}\frac{\partial(U^{\beta})^2}{\partial y} = \tag{3.7}$$

$$F \equiv 2\gamma \arctan(\gamma(x+y-1))/(1+\gamma^2(x+y-1)^2) \text{ in } \Omega = (-\tfrac{1}{2},\tfrac{1}{2}) \times (\tfrac{1}{2},\tfrac{1}{2}) \quad .$$

$$U^{\beta} = \arctan(\gamma(x+y-1)) \text{ on } \partial\Omega \quad ,$$

where $\gamma = 15.8$ is chosen so that the jump from $U(-\frac{1}{48},\frac{1}{48})$ to $U(\frac{1}{48},\frac{1}{48})$ is half the range of U. We use Newton's method, "solving" each step by taking two cycles of the method of Sec. II starting on the finest grid. The Jacobian is updated after each two cycles and L^k recomputed on all grids. More sophisticated updating strategies exist, of course, but the object here is to see if the method can handle this situation, where there is a large jump in the solution over a few mesh intervals. The initial guess is the average of linear interpolation of data from the left and right and the top and bottom boundaries, and the equation to be solved at the $\ell^{\underline{th}}$ step of

the Newton iteration is

$$-\beta h\nabla\cdot(|U^{\beta}_{\ell-1}|\nabla U^{\beta}_{\ell}) + \frac{\partial}{\partial x}(U^{\beta}_{\ell-1}U^{\beta}_{\ell}) + \frac{\partial}{\partial y}(U^{\beta}_{\ell-1}U^{\beta}_{\ell})$$

$$= F + \tfrac{1}{2}\frac{\partial}{\partial x}(U^{\beta}_{\ell})^2) + \tfrac{1}{2}\frac{\partial}{\partial y}((U^{\beta}_{\ell})^2) \quad .$$

The results are summarized in Table 3.9.

TABLE 3.9
Results of method applied to (3.7)

Problem size and parameters	$L'(\Omega)$ error	$\text{Max}(\Omega)$ error	CPU time in seconds on CRAY-1	Linear CF	Nonlinear CF
47x47 (M=5), TOL = 10^{-6} IRELAX=5, IVW=1, β=1	4.6,-2	1.8,-1	1.1	.37	.27
47x47 (M=5), TOL = 10^{-6} IRELAX=5, IVW=1, β=½	2.8,-2	2.3,-1	1.1	.45	.19
47x47 (M=5), TOL = 10^{-6} IRELAX=5, IVW=1, β=¼	3.8,-2	2.9,-1	1.2	.53	.25
47x47 (M=5), TOL = 10^{-6} IRELAX=5, IVW=1, β=1/8	---	---	---	$5.4\text{x}10^{3}$	$2\text{x}10^{17}$
47x47 (M=5), TOL = 10^{-6} IRELAX=5, IVW=2, β=1	4.6,-2	1.8,-1	1.4	.38	.36
47x47 (M=5), TOL = 10^{-6} IRELAX=5, IVW=2, β=½	2.8,-2	2.3,-1	1.2	.25	.15
47x47 (M=5), TOL = 10^{-6} IRELAX=5, IVW=2, β=¼	3.8,-2	2.9,-1	1.2	.48	.14

Finally, we comment that in addition to the numerical evidence presented here, some theoretical investigations have now been performed [B3].

IV. CONCLUSION

In this paper we have extended the results of [D] to a single nonsymmetric equation, exhibiting success with test problem analogous to the ones considered in [D]. The additional problem, raised in [B1], of discontinuous solutions as opposed to discontinuous fluxes of solutions as in [D] has been shown to be amenable to treatment by the technique developed herein.

There are at least two directions for further research. The first is the extension to systems of equations and to more than two space dimensions. The second is to equations with zero dissipation. One way of handling zero dissipation has been discussed in [B1]. There the equations which are relaxed have added to them a healthy amount of dissipation. When their residuals are computed for transfer to the next coarser grid, however, they are computed with zero dissipation. The result is a method which solves such equations (at least smooth model equations) quite nicely to the level of truncation error. Such a double discretization could be implemented into the method of this paper. However, the double discretization method cannot be solved to convergence and practitioners are suspicious - and rightly so - of such a method. Hence, the challenge remains of exhibiting a multigrid method that can solve such equations to algebraic convergence.

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