Multi-Grid Solution of Three-Dimensional Problems With Discontinuous Coefficients

A. Behie
P.A. Forsyth, Jr.

Computer Modelling Group
3512 - 33 Street N.W.
Calgary, Alberta, Canada
T2L 2A6

ABSTRACT

A three-dimensional multi-grid algorithm is developed and compared to an efficient three-dimensional Incomplete Cholesky Conjugate Gradient (ICCG) algorithm on a suite of test problems with discontinouous and/or anisotropic coefficients. The multi-grid algorithm used plane relaxation and a definition of coarse grid operators which is an extension of that used for two-dimensional problems with discontinuous coefficients. Numerical results show that the size of problem for which multi-grid becomes competitive with ICCG is larger than would normally be encountered in reservoir simulation.

INTRODUCTION

Multi-grid methods have proved to be very efficient at solving the two-dimensional diffusion equation with jump discontinuities in the equation coefficients (1,2,3,4,5). These problems are essentially variations of:

$$\nabla \underline{K} \nabla p + \sigma p = -q \tag{1}$$

with strong spatial variation in $\underline{K}$. Multi-grid methods have also been used to solve the non-symmetric convection-diffusion equation in two dimensions (3). These equations are similar to the pressure equation in oil reservoir simulation (6). In reservoir problems the permeability tensor $\underline{K}$ is typically strongly anisotropic and discontinuous.

The method used to solve a two-dimensional version of equation (1) was originally suggested by Alcouffe et al (1). In this report, we generalize these techniques in order to solve equation (1) in three dimensions. Several difficult problems are solved to test the efficiency of this method in three dimensions. The multi-grid solutions are compared with the solutions produced by a D4 ICCG method (7).

MULTI-GRID IN THREE DIMENSIONS

In the following, we assume that the reader is familiar with the basic multi-grid method (8). Let the original equation on the finest grid be denoted by:

$$Lu = f \tag{2}$$

with operator L, unknown u and source term f. If there are M levels in the multi-grid algorithm, then the discretized version of equation (2) on the finest grid is:

$$L^M u^M = f^M \tag{3}$$

For problems with discontinuous coefficients, the operator on the coarse grid is defined recursively (1):

$$L^{K-1} = I_K^{K-1} \; L^K \; I_{K-1}^K$$

$$= (I_{K-1}^K)^T \; L^K \; I_{K-1}^K \qquad (4)$$

where I_{K-1}^K is the interpolation operator from grid K-1 to K. The source term on the coarse grid is given by:

$$f^{K-1} = (I_{K-1}^K)^T \; r^K \qquad (5)$$

where:

$$r^K = f^K - L^K u^K$$

with r^K being the residual on grid K.

The coarse to fine interpolation operator I_{K-1}^K is defined as follows, (see Figure 1) where it is assumed that the fine grid discretization uses at most nearest and next-nearest neighbours (27 point molecule). For fine points corresponding to coarse points, the interpolation is the identity (injection). For fine points on a coarse grid line between two coarse points, the operator L^K is averaged in the two directions perpendicular to the coarse grid line. This averaging amounts to simply adding the components of L^K to produce a collapsed one dimensional operator $(L^K)^{C1}$. The interpolation is then defined by solving:

$$(L^K)^{C1} \; u^K = 0 \qquad (7)$$

Next consider fine points not on coarse grid lines, but in the same plane as two coarse grid lines. These points have four nearest neighbour coarse grid points. The operator L^K is averaged in the direction perpendicular to the plane containing the coarse grid lines. The collapsed operator $(L^K)^{C2}$ is two dimensional, and the interpolation is defined by solving:

$$(L^K)^{C2} \; u^K = 0 \qquad (8)$$

for u^K at the fine point. The remaining fine grid points have eight nearest neighbour coarse grid points, and the original operator L^K can be used to solve for the fine points:

$$L^K \; u^K \; = \; 0 \tag{9}$$

No averaging is required. Note that the coarse grid operators are 27-point (nearest and next-nearest neighbours), even if the fine grid operator has 7 points (nearest neighbours only). This interpolation is a generalization of the methods developed by Alcouffe et al (1).

SMOOTHING METHODS

In order to solve pathological cases such as:

$$p_{xx} + p_{yy} + \varepsilon \, p_{zz} \; = \; 0 \tag{10}$$

with $\varepsilon << 1$, a simple smoothing analysis reveals that plane relaxation in the x-y plane is necessary. This gives a smoothing rate of $5^{-\frac{2}{3}}$ (8). However, the cost of factoring $N^{1/3}$ planes (assuming a cubical region) is:

$$O(N^{1/3} \; (N^{1/3})^2 \; N^{2/3}) \; = \; O(N^{5/3})$$

This set-up cost alone compares unfavourably with ICCG or MICCG (9,10,11,12,13) which require $O(N^{4/3})$ and $O(N^{7/6})$ operations respectively for convergence in three dimensions. In order to avoid this set-up cost, the planes can be solved iteratively with an ICCG method in the planes. This technique will be referred to as incomplete plane relaxation (IPR). The cost of an incomplete LU factorization (ILU) of all the planes is O(N), while the cost of solving the planes to a fixed degree of accuracy is $O(N^{4/3})$ (in the worst case). Consequently, assuming the number of multi-grid cycles required for convergence is independent of N, this algorithm will be $O(N^{4/3})$ in the worst case.

The planes are factored using a third degree (3,7) $\underline{L}\,\underline{D}\,\underline{U}$ incomplete factorization. The compuational molecule is shown in Figure 2, assuming a nine point operator. The element marked D is the diagonal, while points corresponding to the bands in $\underline{L}$ are labelled by L with a subscript giving the band number. Similarly with the bands in $\underline{U}$. For convenience, the same template is used on the finest grid (which has a five point operator in the plane). The bands marked L_1, U_1 are absent in this case. On the finest grid, this decomposition is between fourth and fifth degree.

An alternate smoothing method using 3D ILU factorization was coded and tested. The work for a fixed number of 3D ILU sweeps on each grid is O(N) per multi-grid cycle. Thus, a multi-grid algoritym based on this smoothing is also O(N) (assuming the number of cycles to reduce the

residual by a fixed amount is O(1)). However one knows, a priori, that such an algorithm will not work well on pathological problems of the type considered in equation (10).

The 3D ILU factorization used was first degree on the coarse grids, having twenty-seven bands corresponding to the twenty-seven bands of the differential operator. On the fine grid the same twenty-seven point template was used but the factorization was now between second and third degree, since the differential operator on the fine grid had only seven points.

COMPARISON METHODS

The performance of the 3D multi-grid algorithm was compared to that of the 3D Incomplete Cholesky Conjugate Gradient (ICCG) (9,10) algorithm. The work per iteration for ICCG methods is O(N), while the number of iterations to reduce the residual by a fixed amount is $O(N^{1/d})$ where d is the number of spatial dimensions. This means that the 3D ICCG algorithm is $O(N^{4/3})$ which implies that it is formally of the same order as multi-grid with IPR. Note that the order of 3D ICCG is lower than that of 2D ICCG.

The particular preconditioning used here was a first degree, D4 ordered incomplete LU (ILU) factorization. This preconditioning, with ORTHOMIN acceleration, has been found to be optimal for a wide range of nonsymmetric model and reservoir simulation problems (7,14) in both two and three spatial dimensions.

TEST PROBLEMS

The test problem used was the three-dimensional problem:

$$\frac{\partial}{\partial x}\left(KX\,\frac{\partial p}{\partial x}\right) + \frac{\partial}{\partial y}\left(KY\,\frac{\partial p}{\partial y}\right) + \frac{\partial}{\partial z}\left(KZ\,\frac{\partial p}{\partial z}\right)$$

$$-\sigma p = -q \qquad (11)$$

which was discretized in the following way on the finest grid:

$$\frac{KX_{i+\frac{1}{2},j,k}}{h^2}\left(p_{i+1,j,k} - p_{i,j,k}\right) - \frac{KX_{i-\frac{1}{2},j,k}}{h^2}\left(p_{i,j,k} - p_{i-1,j,k}\right)$$

$$+\frac{KY_{i,j+\frac{1}{2},k}}{h^2}\left(p_{i,j+1,k} - p_{i,j,k}\right) - \frac{KY_{i,j-\frac{1}{2},k}}{h^2}\left(p_{i,j,k} - p_{i,j-1,k}\right)$$

$$+\frac{KZ_{i,j,k+\frac{1}{2}}}{h^2}(p_{i,j,k+1}-p_{i,j,k})-\frac{KZ_{i,j,k-\frac{1}{2}}}{h^2}(p_{i,j,k}-p_{i,j,k-1})$$

$$-\frac{\sigma_{i,j,k}}{h^2}\,p_{i,j,k}=\frac{-q_{i,j,k}}{h^2} \tag{12}$$

with

$$p_{i,j,k} = p\,(ih, jh, kh)$$

$$\sigma_{i,j,k} = \sigma\,(ih, jh, kh)h^2$$

$$q_{i,j,k} = q\,(ih, jh, kh)h^2$$

Terms such as $KX_{i+\frac{1}{2},j,k}$, $KY_{i,j+\frac{1}{2},k}$ and so on, are given by the harmonic mean to ensure continuity of fluid flux (6).

All problems were solved on the unit cube with

$$NX = NY = NZ$$

where NX, NY, and NZ are the number of grid nodes in the x-, y- and z-directions respectively.

For problems #1, #3, #5, #7, and #9 there were unit sources of opposite sign at (1,1,1) and (NX, NY, NZ) and a compressibility $\sigma = 0$.

For problems #2, #4, #6, #8, and #10 there was a unit source at (1,1,1) and a compressibility of $\sigma = 10^{-4}$.

All problems had Neumann boundary conditions. Both 9x9x9 and 17x17x17 problems were solved.

Problems #1 and #2

The geometry of these problems is described by:

$$KX = KY = KZ = K$$

and $K = 0.001$ for

$$i_1 \le i \le i_2$$

$j_1 \leq j \leq j_2$

$k_1 \leq k \leq k_2$

otherwise K = 1.0.

For the 9x9x9 case

$i_1 = j_1 = k_1 = 4$

and $i_2 = j_2 = k_2 = 6$

and for the 17x17x17 case

$i_1 = j_1 = k_1 = 7$

and $i_2 = j_2 = k_2 = 11$

Problems #3 and #4

These problems can be described by

KX = KY = 1.0

KZ = 0.001

for $1 \leq i \leq NX$

$1 \leq j \leq NY$

$1 \leq k \leq NZ$

Problems #5 and #6

These problems can be described by

KX = KY = KZ = K

and $K = 10^{-3}$

for $1 \leq i \leq i_1$

$1 \leq j \leq j_1$

$1 \leq k \leq k_1$

$K = 10^3$

for $i_2 \leq i \leq NX$

$j_2 \leq j \leq NY$

$k_2 \leq k \leq NZ$

and $K = 1$ otherwise.

For 9x9x9 case

$i_1 = j_1 = k_1 = 4$

and $i_2 = j_2 = k_2 = 6$

For the 17x17x17 case

$i_1 = j_1 = k_1 = 7$

and $i_2 = j_2 = k_2 = 11$

Problems #7 and #8

The geometry of these problems is given by

$KX = 1$

$KY = 10^3$

$KZ = 10^{-3}$

for $1 \leq i \leq NX$

$1 \leq j \leq NY$

$1 \leq k \leq NZ$

Problems #9 and #10

In these problems

$KX = 1$

$KY = 10^2$

$KZ = 10^{-2}$

for

$$1 \le i \le i_1$$

$$1 \le j \le j_1$$

$$1 \le k \le k_1$$

and

$$KX = 10^{-2}$$

$$KY = 1$$

$$KZ = 10^{2}$$

for $i_1 < i \le NX$

$j_1 < j \le NY$

$k_1 < k \le NZ$

For the 9x9x9 case

$$i_1 = j_1 = k_1 = 4$$

and for the 17x17x17 case

$$i_1 = j_1 = k_1 = 7$$

COMPUTATIONAL DETAILS AND RESULTS

The multi-grid algorithm with IPR as a smoother was run with 1 sweep of x-y planes followed by x-z planes followed by y-z planes on each grid. The convergence criterion in each plane was:

$$|| r^p ||_\infty < || r_0^p ||_\infty \times 10^{-5} \quad (13)$$

or

$$|| r^p ||_\infty < || r_0 ||_\infty \times 10^{-6} \quad (14)$$

where $|| \; ||_\infty$ denotes the ℓ_∞ norm, r^p denotes the residual in the plane, r_0^p denotes the residual in the plane after 1 ICCG iteration, and r_0 denotes the initial residual over all grid nodes.

The multi-grid algorithm with 3D ILU smoothing was run with 1 sweep of ILU on each grid. Both multi-grid algorithms had an exact solve on the coarsest grid. The outer convergence tolerance for both versions of the multi-grid algorithm and for the 3D ICCG algorithm was

$$|| r ||_\infty < || r_0 || \times 10^{-6} \qquad (15)$$

where $|| \; ||_\infty$ is as above, r is the residual over all nodes and r_0 is the initial residual over all nodes. The initial guess was zero for all algorithms. The 9x9x9 problems used three multi-grid levels. The 17x17x17 problems used four.

Table 1 gives the results for test problems 1 to 10 for the multi-grid algorithm with IPR smoothing (for 9x9x9 and 17x17x17 problems) and for the multi-grid algorithm with 3D ILU smoothing (for the 9x9x9 case only). Table 2 gives the results for test problems 1 to 10 for the 3D ICCG algorithm.

The table entries are operation counts (multiplications and divisions) in terms of a work unit defined by N operations (N being the number of grid nodes on the finest grid). They include only iteration work and not set-up work. The number of multi-grid cycles and number of ICCG iterations are given in brackets. Absence of entries indicates that specified CPU time limit was exceeded.

For disucssion of results the test problems can be divided into three classes

(i) discontinuous problems (problems #1, #2, #5, and #6)

(ii) anisotropic problems (problems #3, #4, #7, and #8)

(iii)anisotropic and discontinuous problems (problems #9 and #10)

From Table 1 we observe that, as expected, 3D ILU smoothing performed poorly on problems with anisotropies. On problems with discontinuities it converged more slowly than the algorithm with IPR smoothing, but the lower work count per cycle made it a competitive method.

Table 1 also shows that the mulit-grid algorithm with IPR smoothing performs well on discontinuous (class (i)) problems, taking only a few cycles for convergence. However, the large amount of work to solve all the planes to an adequate degree of accuracy increases the total work count substantially. This algorithm also handles the class (ii) and class (iii) problems with no compressibility (σ=0) adequately. The class (ii) and class (iii) problems with small compressibility, on the other hand, are handled very poorly. This result is surprising and we do not at this time have an explanation for it. We have tested the same problems with a two level multi-grid scheme with similar results.

Note that as the size of the problem is doubled, multi-grid with IPR has the same work count. Thus the algorithm in O(N). Recall that in Section 3 we showed that it was formally $O(N^{4/3})$. This discrepancy can be explained by noting that there were two conditions (given by equations (13) and (14)) governing the number of ICCG sweeps in a plane. Adherance to condition (13) gives the $O(N^{4/3})$ property, since the number of ICCG sweeps in a plane is related to the reduction of the residual by a fixed amount. In practice the algorithm must be satisfying condition (14) first, and thus doing only a few ICCG sweeps in each plane. This fact has been confirmed by actually printing the number of ICCG sweeps in each plane for a number of cases. Note also that the algorithm has been run in such a way as to force the plane ICCG to converge to very tight tolerances. This did not affect the number of multi-grid cycles required for convergence, nor did it alleviate the difficulty with class (ii) and (iii) problems with small compressibility.

Table 2 illustrates the performance of the 3D ICCG algorithm described in Section 4. For the 9x9x9 problems ICCG is always superior to multi-grid. For the 17x7x17 problems this is again true, with case 7 being the exception. Thus we conclude that the size of problem for which multi-grid begins to compete with ICCG is certainly larger than 17x17x17. Extrapolation of the values in Tables 1 and 2 indicates that the crossover point will probably be larger than 33x33x33, which is currently larger than would normally be encountered in reservoir simulation.

It is interesting to note that the class (ii) and class (iii) problems with zero compressibility were fairly difficult for the ICCG algorithm and that the degree of difficulty correlated to the amount of anisotropy (problem #7 being more difficult than problem #3, for instance). The multi-grid algorithm seemed to find them of the same degree of difficulty, taking 8 multi-grid cycles for convergence (for the 9x9x9 case).

Table 2 also shows that 3D ICCG is not an O(N) algorithm. We expect from theory that the work should increase as

$$\sim (2.2.2)^{1/3} = 2 \qquad (16)$$

In practice, we do not see this ratio but a range from 1.2 to 3.3. This could be attributed to the presence of a large number of boundary nodes in the 9x9x9 case.

CONCLUSIONS

A three-dimensional multi-grid algorithm has been developed and compared to a very efficient three-dimensional ICCG algorithm.

The multi-grid algorithm performed well for problems with discontinuous coefficients and either zero or small compressibilities. For problems with discontinuous and/or anisotropic coefficients and zero compressibilities the multi-grid algorithm also performed well. For

problems with discontinuous and/or anisotropic coefficients and small compressibilities anomalous results were obtained. Note that the same results were obtained with a two level scheme.

Multi-grid with three-dimensional ILU as a smoother was adequate for problems with discontinuous coefficients but not for those with anisotropic coefficients.

The three-dimensional multi-grid algorithm although not formally of O(N) was found to be O(N) on all the problems tested. The three-dimensional ICCG algorithm was not O(N).

The size of problem for which three-dimensional multi-grid is likely to be a viable solution option is larger than would normally be encountered in practical reservoir simulation.

It would appear that further investigation of the interpolation used in the three-dimensional algorithm should be carried out in order to improve the method for the anomalous cases.

ACKNOWLEDGEMENTS

This research was supported by the Associate Members of CMG and by the Alberta/Canada Energy Resources Research Fund administered by the Department of Energy and Natural Resources of the Province of Alberta.

NOMENCLATURE

$\underline{D}$ - diagonal matrix

f - source term in differential equation

h - grid spacing

I^K_{K-1} - interpolation operator for grid K-1 to grid K

$\underline{K}$, KX-KY,KZ - discontinuous coefficients in test problems (permeabilities)

L - differential operator

$\underline{L}$ - lower triangular matrix

N - number of grid nodes (=NX * NY * NZ) on finest grid

NX - number of grid nodes in x-direction

NY - number of grid nodes in y-direction

NZ - number of grid nodes in z-direction

O - order

p - pressure

q - source term in test problems

r, r^p - residual

u - solution

$\underline{U}$ - upper triangular matrix

σ - compressibility term in test problems

Subscripts and Indices

i,j,k - grid point labels

Superscripts

C_1 - operator collapsed to one spatial dimension

C_2 - operator collapsed to two spatial dimensions

K - grid level

p - plane

T - transpose

Symbols

$||\ ||_\infty$ ℓ_∞ norm

REFERENCES

1. R.E. Alcouffe, A. Brandt, J.E. Dendy, Jr., and J.W. Painter, "The Multi-grid Method for the Diffusion Equation with Strongly Discontinuous Coefficients", SIAM J. Sci. Stat. Comp., 2, (1981), 430-454.

2. J.E. Dendy, Jr., "Black Box Multi-grid", Symposium on Multi-grid Methods, NASA Ames Research Centre, 1981.

3. A. Behie and P. Forsyth, Jr., "Comparison of Fast Iterative Methods for Symmetric Systems", Inst. Math. Applic. J. Num. Anal. (in press).

4. A. Behie and P. Forsyth, Jr., "Multi-grid Solution of the Pressure Equation in Reservoir Simulation", paper SPE 10492, presented at the Sixth Symposium on Reservoir Simulation, New Orleans, 1982 (submitted to Soc. Pet. Eng. J.).

5. R. Kettler and J.A. Meijerink, "A Multi-grid Conjugate Gradient Method for Elliptic Problems with Strongly Discontinuous Coefficients in General Domains", Shell Research Report 604, Rijswijk, the Netherlands, 1981.

6. K. Aziz and A. Settari, Petroleum Reservoir Simulation, Applied Science, London, 1979.

7. A. Behie and P. Forsyth, Jr., "Incomplete Factorization Methods for Fully Implicit Simulation of Enhanced Oil Recovery", presented at the Sparse Matrix Symposium, Fairfield Glade, Tennessee, 1982 (submitted to SIAM J. Sci. Stat. Comp.).

8. A. Brandt, "Multi-Level Adaptive Solutions to Boundary-Value Problems", Math. Comp., 31, (1977), 333-390.

9. J.A. Meijerink and H.A. van der Vorst, "An Iterative Solution Method for Linear Systerms in Which the Coefficient Matrix is a Symmetric M-Matrix", Math. Comp., 31, (1977), 148-162.

10. D.S. Kershaw, "The Incomplete Cholesky Conjugate Gradient Method for the Iterative Solution of Systems of Linear Equations", J. Comp. Phys., 26, (1978), 43-65.

11. I. Gustafsson, "A Class of First Order Factorization Methods", Computer Sciences 77.04 R, Chalmers University of Technology, Goteborg, Sweden (1977).

12. I. Gustafsson, "On First Order Factorization Methods for the Solution of Problems with Discontinuous Material Coefficients", Computer Sciences 77.13 R, Chalmers University of Technology, Goteborg, Sweden (1977).

13. J.M. Hyman, "High Order Incomplete Factorizations of Sparse Matrices", Los Alamos Report LA-UR-80-3701 (1980).

14. D. Collins and P. Forsyth, Jr., "Further Developments in Incomplete Factorization Methods", CMG.R6.06.

Table 1

Work Count* of Multi-Grid Algorithm
With Two Different Smoothing Methods

Problem	IPR		3D ILU
	9x9x9	17x17x17	9x9x9
#1	927 (4)	870 (4)	740 (9)
#2	675 (3)	620 (3)	740 (9)
#3	2,525 (8)	2,542 (8)	7,921 (101)
#4	5,042 (15)	3,476 (10)	8,055 (101)
#5	1,280 (6)	1,041 (5)	818 (10)
#6	1,056 (5)	796 (4)	818 (10)
#7	1,774 (8)	1,587 (7)	7,920 (101)
#8	14,073 (57)	10,668 (48)	--
#9	3,357 (8)	4,426 (9)	7,920 (101)
#10	43,385 (101)	--	--

*One work unit = N operations
Number of cycles/iterations in brackets

Table 2

Work Count* of ICCG Algorithm

Problem	9x9x9	17x17x17
#1	223 (10)	355 (16)
#2	201 (9)	245 (11)
#3	707 (32)	1,807 (82)
#4	773 (35)	1,983 (90)
#5	179 (8)	399 (18)
#6	311 (14)	641 (29)
#7	1,345 (61)	3,127 (142)
#8	1,565 (71)	5,283 (240)
#9	839 (38)	1,609 (73)
#10	949 (43)	1,785 (81)

*One work unit = N operations. Numbers of cycles/iterations in brackets

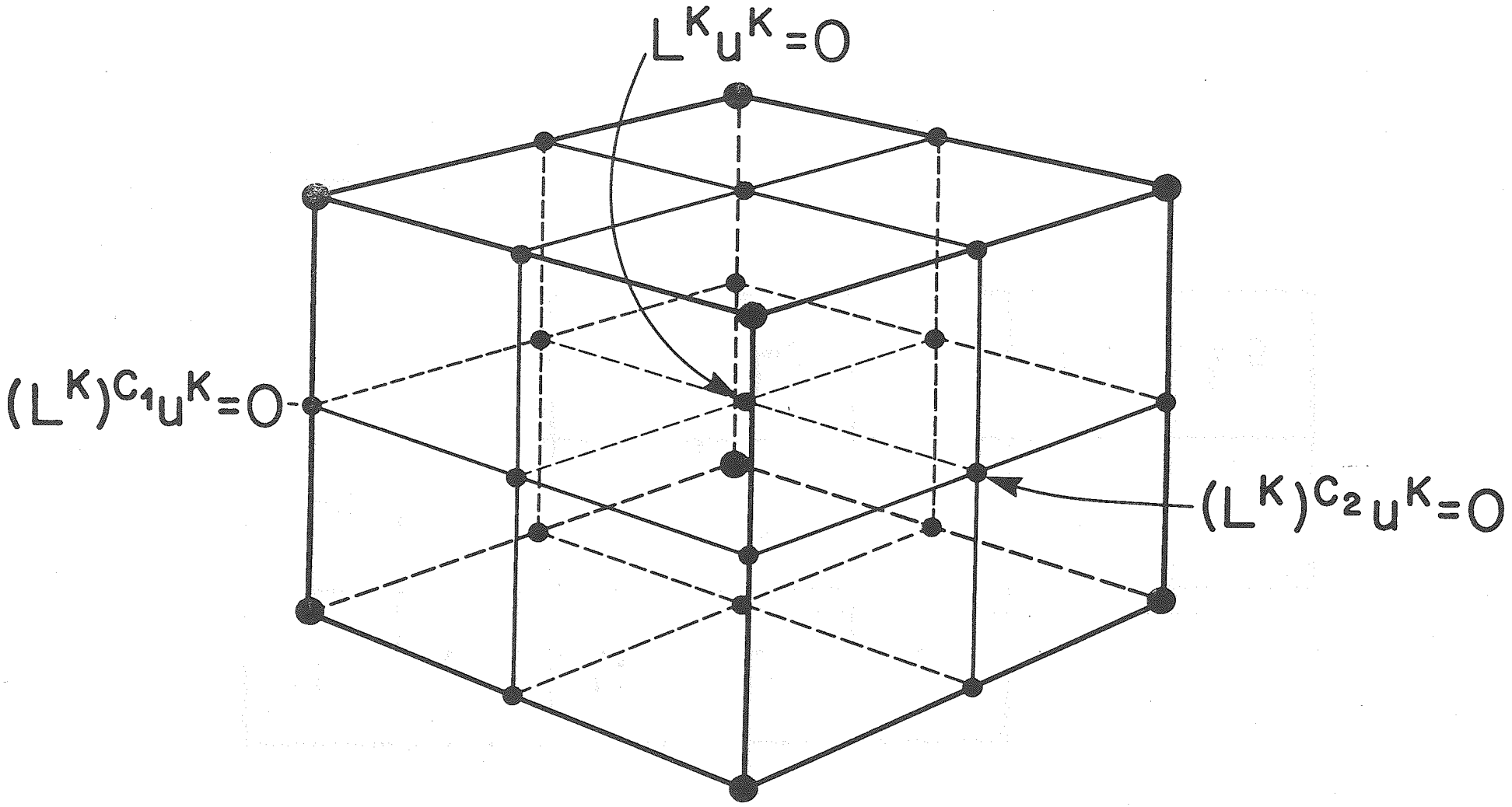

FIGURE 1. Coarse to Fine Interpolation Operator

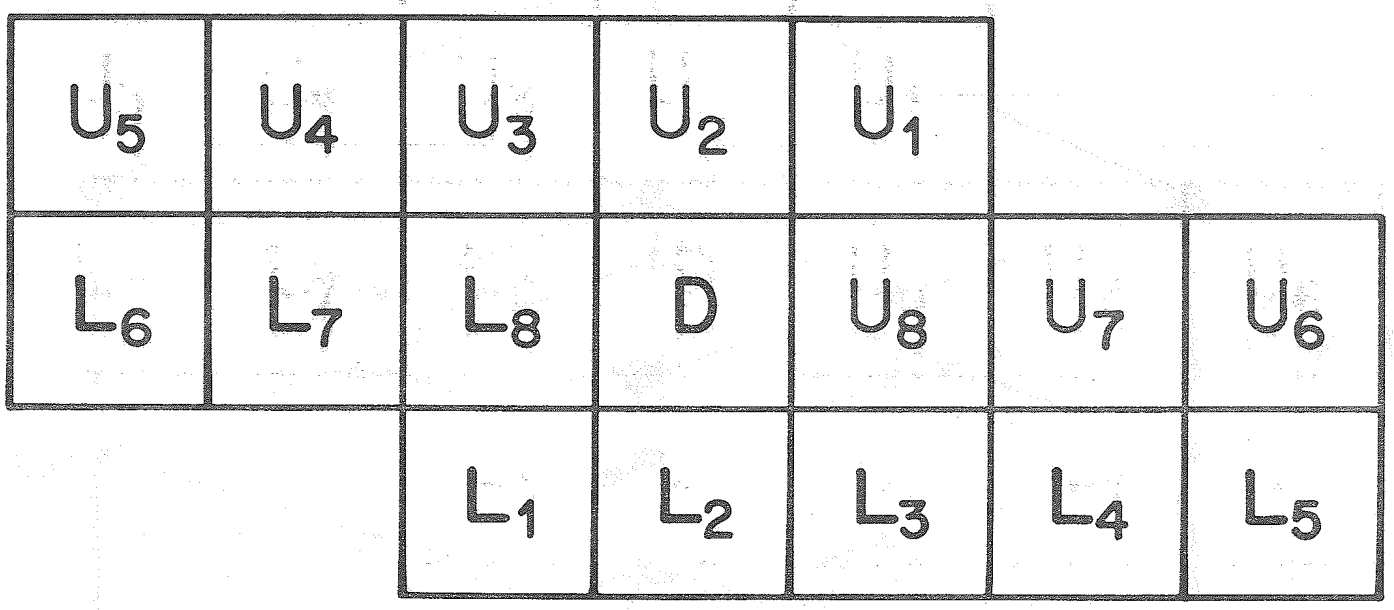

FIGURE 2. Computational Molecule For Plane Factorization