Higher-Order Multigrid Methods for the Solution of Elliptic Equations

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Multigrid Method – Basic Outline

- Consider the solution of difference equation

$$L^h u^h = f^h \qquad (1)$$

- Let U^h be an approximation to u^h and V^h the correction $(u^h - U^h)$

- Consider a sequence of grids h, 2h, 4h, etc.

  On a grid of spacing 2h, replace Eq. (1) by

$$L^{2h} V^{2h} + I_h^{2h} L^h U^h = f^{2h} \qquad (2)$$

  where I_h^{2h} is restriction operator

- Compute V^{2h} and improve the solution on grid h by

$$U^h_{new} = U^h_{old} + \bar{I}^h_{2h} V^{2h} \qquad (3)$$

  where $\bar{I}^h_{2h}$ is interpolation operator.

FULL APPROXIMATION SCHEME (FAS)

IF L IS A NONLINEAR OPERATOR

- REPLACE THE COARSE GRID EQ. (2) BY

$$L^{2H} \bar{U}^{2H} = L^{2H} I_H^{2H} U^H + I_H^{2H} \left(F^H - L^H U^H \right)$$

- REPLACE UPDATING EQ. (3) BY

$$U^H_{NEW} = U^H_{OLD} + I^H_{2H} \left(\bar{U}^{2H} - I_H^{2H} U^H_{OLD} \right)$$

HIGHER-ORDER MULTIGRID METHODS

HIGHER-ORDER-ACCURATE MULTIGRID SOLUTION STRATEGIES CAN BE DEVISED BY EMPLOYING

- HIGHER-ORDER DIFFERENCE OPERATORS L^H
- HIGHER-ORDER INTERPOLATION OPERATOR I_{2H}^{H}
- HIGHER-ORDER-ACCURATE COARSE GRID CORRECTION

Main elements of multigrid method

(1) A smoothing operator to efficiently reduce the high-frequency errors on a given grid

(2) A restriction operator to transfer the solution from a fine grid to a coarse grid and to provide proper coarse-grid correction

(3) A prolongation operator to interpolate the solution from coarse to fine grid

Rapid solution procedure for two reasons:

(1) Number of operations required for a relaxation sweep on one of the coarse grids is much smaller than number required on the fine grid.

(2) The rate of convergence is faster on a coarse grid, since the corrections can be propagated from one end of the grid to the other in a small number of steps.

FOURTH-ORDER-ACCURATE MULTIGRID SOLUTION STRATEGIES

(1) METHOD OF τ-EXTRAPOLATION

- TRUNCATION ERROR τ^H IS ASSUMED TO HAVE THE LOCAL EXPANSION

  $$\tau^H \equiv L_2^H U - F^H = AH^2 + O(H^4)$$

- EXTRAPOLATION TO GRID '2H' GIVES

  $$\tau_H^{2H} = \frac{4}{3}\left(\tau^{2H} - \tau^H\right) + O(H^4)$$

- THUS, ON GRID '2H', FOURTH-ORDER ACCURACY IS OBTAINED BY SOLVING THE EQUATION

  $$L_2^{2H}\,\bar{U}^{2H} = F^{2H} + \tau_H^{2H} \qquad (1)$$

WHERE

$$\tau_H^{2H} = \frac{4}{3}\left[\left(L_2^{2H}\, I_H^{2H}\, U^H - F^{2H}\right) - I_H^{2H}\left(L_2^H\, U^H - F^H\right)\right]$$

- SOLVE EQ. (1) ON GRID '2H' BY FAS CYCLE.

- USE CUBIC INTERPOLATION OPERATOR I_{2H}^H TO OBTAIN THE SOLUTION ON THE FINEST GRID 'H'.

(2) METHOD OF ITERATIVE IMPROVEMENT

- SOLVE $L_2^H U^H = F^H$ USING FAS CYCLE

- SOLVE $L_2^H V^H = 2F^H - L_4^H U^H$

  OR $L_2^H V^H = F^H + L_2^H U^H - L_4^H U^H$

  USING FAS CYCLE

- V^H HAS FOURTH-ORDER ACCURACY

- TWO SOLUTIONS ARE NEEDED USING SECOND-ORDER OPERATOR L_2^H.

(3) FOURTH-ORDER RELAXATION OPERATOR ON THE FINEST MESH AND SECOND-ORDER OPERATOR ON COARSER MESHES

- MAKE A FEW RELAXATION SWEEPS ON $L_4^H U^H = F^H$
- SOLVE $L_2^{2H} \overline{U}^{2H} = L_2^{2H} I_H^{2H} U^H + I_H^{2H} (F^H - L_4^H U^H)$ USING FAS CYCLE
- INTERPOLATE THE SOLUTION TO GRID H AND MAKE A RELAXATION SWEEP ON $L_4^H U^H = F^H$.

(4) FOURTH-ORDER RELAXATION OPERATOR ON ALL GRIDS

- SOLVE $L_4^H U^H = F^H$ BY AN FAS CYCLE
- COARSE GRID EQUATION TAKES THE FORM

$$L_4^{2H} \overline{U}^{2H} = L_4^{2H} I_H^{2H} U^H + I_H^{2H} (F^H - L_4^H U^H)$$

- UPDATING EQUATION REMAINS THE SAME

$$U_{NEW}^H = U_{OLD}^H + I_{2H}^H (\overline{U}^{2H} - I_H^{2H} U_{OLD}^H)$$

Higher-Order-Accurate Finite-Difference Methods

- Standard Higher-Order-Accurate Discretization
- Mehrstellung or Compact Scheme
- Hodie Method (Higher-Order Difference Approximation with Identity Expansion)
- Spline-Collocation Method

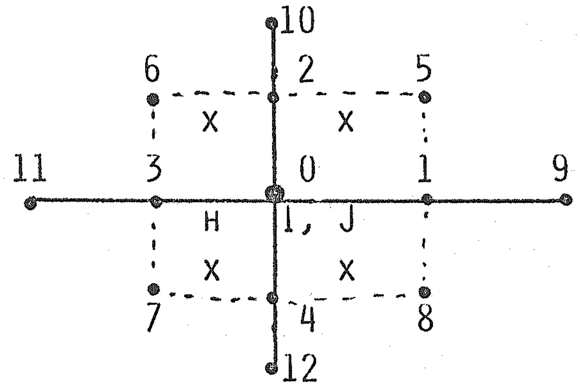
Fourth-Order Finite-Difference Methods - Basic Concepts

- Consider the solution of the elliptic equation

$$Lu = A\,u_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G, \quad AC > B^2 \text{ in } \Omega$$

$$u = g \text{ on } \partial\Omega$$

- Finite-difference stencil



- Standard- differencing

$$(U_{xx})_{I,J} = \frac{(-U_{I-2,J} + 16U_{I-1,J} - 30\,U_{I,J} + 16U_{I+1,J} - U_{I+2,J})}{12h^2}$$

$$(U_x)_{I,J} = \frac{-U_{I+2,J} + 8U_{I+1,J} - 8U_{I-1,J} + U_{I-2,J}}{12h}$$

$$L^h\,U^h \equiv \frac{1}{h^2}\sum_{I=0}^{4} \alpha_I\,U_I + \frac{1}{h^2}\sum_{I=9}^{12} \alpha_I U_I = G_0^h$$

- COMPACT-DIFFERENCING

U AND THE DERIVATIVE U_X, U_Y, U_{XX}, U_{YY} ARE CONSIDERED UNKNOWN

$$\frac{1}{6}(U_X)_{I+1,J} + \frac{2}{3}(U_X)_{I,J} + \frac{1}{6}(U_X)_{I-1,J} = \frac{1}{2H}(U_{I+1,J} - U_{I-1,J})$$

$$\frac{1}{12}(U_{XX})_{I+1,J} + \frac{5}{6}(U_{XX})_{I,J} + \frac{1}{12}(U_{XX})_{I-1,J} = \frac{1}{H^2}(U_{I+1,J} - 2U_{I,J} + U_{I-1,J})$$

$$L^H U^H \equiv \frac{1}{H^2}\sum_{I=0}^{8} \alpha_I U_I = \sum_{I=0}^{8} \beta_I G_I \equiv R^H G^H$$

I'S ARE THE STENCIL POINTS

- HODIE SCHEME

$$L^H U^H \equiv \frac{1}{H^2} \sum_{I=0}^{8} \alpha_I U_I = \sum_{J=1}^{8} \beta_J G_J \equiv \bar{R}^H G^H$$

I'S ARE THE STENCIL POINTS

J'S ARE THE EVALUATION POINTS WHICH MAY BE DIFFERENT THAN THE STENCIL POINTS

α_I AND β_J ARE DETERMINED BY REQUIRING THAT THE DIFFERENCE APPROXIMATION BE EXACT ON SOME FINITE-DIMENSIONAL LINEAR SPACE S, SUCH AS SPACE P_M OF POLYNOMIALS OF DEGREE ATMOST M. THE DIMENSION OF THE SPACE S IS

$$\text{DIM}(S) = K + 1 = (M + 1)(M + 2) / 2$$

FOR ANY BASIS $s_0 \cdot \cdot \cdot \cdot s_K$ OF S, α_I AND β_J SATISFY

$$\frac{1}{H^2} \sum_{I=0}^{8} \alpha_I (s_K)_I = \sum_{J=1}^{J} \beta_J (Ls_K)_J , \quad K = 0, \ldots\ldots K$$

USUALLY $J = \text{DIM}(S) - 8$ WITH $\beta_0 = 1$

FOR A FOURTH-ORDER SCHEME $J = 13$

2-D Poisson Equation

- $-\Delta U = F$

- Standard difference equation (second-order-accurate)

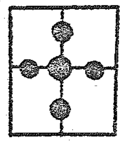
$$L_2^H U^H = \frac{1}{H^2} \begin{bmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{bmatrix} U = -F_0^H$$

- Compact difference equation (fourth-order-accurate)

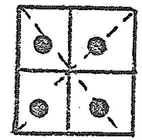
$$L_4^H U^H \equiv \frac{1}{H^2} \begin{bmatrix} 1 & 4 & 1 \\ 4 & -20 & 4 \\ 1 & 4 & 1 \end{bmatrix} U = -\frac{1}{24} \begin{bmatrix} 1 & 10 & 1 \\ 10 & 100 & 10 \\ 1 & 10 & 1 \end{bmatrix} F^H$$

- Hodie difference equation (fourth-order-accurate)

Consider the following sets of evaluation points for 'F':



$P(H,\xi)$



$Q(H,\xi)$

$0 < \xi \leq 1$

(A) one-parameter family of P-approximation

$$L_4^H U^H = \frac{1}{H^2}\begin{bmatrix} 1 & 4 & 1 \\ 4 & -20 & 4 \\ 1 & 4 & 1 \end{bmatrix} = -\frac{1}{2\xi^2}\begin{bmatrix} & \bullet 1 & \\ 1 \bullet & 4(3\xi^2-1) & \bullet 1 \\ & \bullet 1 & \end{bmatrix} F^H = -\frac{1}{2}\begin{bmatrix} & 1 & \\ 1 & 8 & 1 \\ & 1 & \end{bmatrix} F^H \text{ if } \xi = 1$$

(B) One-parameter family of Q-approximation

$$L_4^H U^H \equiv \frac{1}{H^2}\begin{bmatrix} 1 & 4 & 1 \\ 4 & -20 & 4 \\ 1 & 4 & 1 \end{bmatrix} = -\frac{1}{4\xi^2}\begin{bmatrix} 1 \bullet & & \bullet 1 \\ & 4(6\xi^2-1) & \\ 1 \bullet & & \bullet 1 \end{bmatrix} F^H = \frac{1}{4}\begin{bmatrix} 1 & & 1 \\ & 20 & \\ 1 & & 1 \end{bmatrix} \text{ if } \xi = 1$$

SMOOTHING FACTOR $\bar{\mu}$ FOR 2-D POISSON EQUATION

- STANDARD FIVE-POINT GAUSS-SEIDEL RELAXATION

$$\bar{\mu}_S(\theta_1, \theta_2) = \left| \frac{E^{I\theta_1} + E^{I\theta_2}}{4 - E^{-I\theta_1} - E^{-I\theta_2}} \right|,$$

$$\text{MAX } \bar{\mu}_S = \bar{\mu}_S\left(\frac{\pi}{2}, \cos^{-1}\frac{4}{5}\right) = 0.5$$

- COMPACT NINE-POINT GAUSS-SEIDEL RELAXATION

$$\bar{\mu}_C(\theta_1, \theta_2) = \left| \frac{4\left(E^{I\theta_1} + E^{I\theta_2}\right) + E^{I(\theta_1+\theta_2)} + E^{I(\theta_1-\theta_2)}}{20 - 4\left(E^{-I\theta_1} + E^{-I\theta_2}\right) - E^{-(\theta_1+\theta_2)} - E^{I(\theta_1-\theta_2)}} \right|,$$

$$\text{MAX } \bar{\mu}_C = \bar{\mu}_C\left(\frac{\pi}{2}, \frac{3\pi}{16}\right) = 0.464$$

Numerical Example

$$-\Delta u = f = \sin 3(x + y) \text{ in } \Omega = (0, 3) \times (0, 2)$$

$$u = g = \cos 3(x + y) \text{ on } \partial\Omega$$

Finest mesh = 97 x 65 with spacing $h = 1/32$

No. of grids = 6

Fixed cycling MG - Algorithm

2*6, 2*5, 2*4, 2*3, 2*2, 4*1, 1*2, 1*3, 1*4, 1*5, 1*6

Relative Error in L_2-Norm

Method	Error
Second-order	0.41 (-3)
τ	0.27 (-4)
Iterative improvement-c	0.32 (-6)
HF (compact)	0.26 (-6)
H (compact)	0.11 (-6)
HF (Hodie)-P	0.53 (-6)
H (Hodie)-P	0.12 (-6)
HF (Hodie)-Q	0.09 (-5)
H (Hodie)-Q	0.78 (-6)

2-D HELMHOLTZ EQUATION

- $-\Delta U + CU = F \quad , \; C \geq 0$

- STANDARD DIFFERENCE EQUATION (SECOND-ORDER-ACCURATE)

$$L_2^H U^H \equiv \frac{1}{H^2} \begin{bmatrix} & 1 & \\ 1 & -(4+C) & 1 \\ & 1 & \end{bmatrix} U = -F_0^H$$

- COMPACT DIFFERENCE EQUATION (FOURTH-ORDER-ACCURATE)

$$L_4^H U^H \equiv \frac{1}{H^2} \begin{bmatrix} \alpha_6 & \alpha_2 & \alpha_5 \\ \alpha_3 & \alpha_0 & \alpha_1 \\ \alpha_7 & \alpha_4 & \alpha_8 \end{bmatrix} = -\frac{1}{24} \begin{bmatrix} 1 & 10 & 1 \\ 10 & 100 & 10 \\ 1 & 10 & 1 \end{bmatrix} F$$

$$\alpha_0 = -20 - \frac{25}{6} CH^2$$

$$\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 4 - \frac{5}{12} CH^2$$

$$\alpha_5 = \alpha_6 = \alpha_7 = \alpha_8 = 1 - \frac{1}{24} CH^2$$

- HODIE DIFFERENCE EQUATION (FOURTH-ORDER-ACCURATE)

(A) ONE-PARAMETER FAMILY OF P-APPROXIMATION ($0 < \xi < 1$).

$$L_4^H U^H \equiv \frac{1}{H^2}\begin{bmatrix} \alpha_6 & \alpha_2 & \alpha_5 \\ \alpha_3 & \alpha_0 & \alpha_1 \\ \alpha_7 & \alpha_4 & \alpha_8 \end{bmatrix} = -\frac{1}{2\xi^2}\begin{bmatrix} & 1\bullet & \\ 1\bullet & \beta_0 & \bullet 1 \\ & 1\bullet & \end{bmatrix} F$$

$$\alpha_0 = -20 - 2(3 - \xi^2)\, CH^2 - (1 - \xi^2)\, C^2H^4/2$$

$$\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 4 - \xi^2\, CH^2/2$$

$$\alpha_5 = \alpha_6 = \alpha_7 = \alpha_8 = 1$$

$$\beta_0 = 4(3\xi^2 - 1) + \xi^2(1 - \xi^2)\, CH^2$$

(B) ONE-PARAMETER FAMILY OF Q-APPROXIMATION ($0 < \xi \leq 1$).

$$L_4^H U^H \equiv \frac{1}{H^2} \begin{bmatrix} \alpha_6 & \alpha_2 & \alpha_5 \\ \alpha_3 & \alpha_0 & \alpha_1 \\ \alpha_7 & \alpha_4 & \alpha_8 \end{bmatrix} = -\frac{1}{4\xi^2} \begin{bmatrix} 1 \bullet & & \bullet 1 \\ & \beta_0 & \\ 1 \bullet & & \bullet 1 \end{bmatrix} F$$

$$\alpha_0 = -20 - (6 - \xi^2)\, CH^2 - (1 - \xi^2)\, C^2H^2/2$$

$$\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 4$$

$$\alpha_5 = \alpha_6 = \alpha_7 = \alpha_8 = 1 - \xi^2\, CH^2/4$$

$$\beta_0 = 4(6\xi^2 - 1) + 2\xi^2(1 - \xi^2)\, CH^2$$

Numerical Example

$$-\Delta u + Cu = f = \sin 3(x + y) \text{ in } \Omega = (0, 3) \times (0, 2)$$

$$u = g = \cos 3(x + y) \text{ on } \partial\Omega$$

Finest mesh = 97 x 65 with spacing $h = 1/32$
No. of grids = 6
Fixed cycling MG - algorithm

Relative Error in L_2-Norm ($C = 2$)

Method	Error
Second-order	0.29 (-2)
τ	0.18 (-3)
Iterative improvement-C	0.17 (-4)
HF (compact)	0.09 (-4)
H (compact)	0.03 (-4)
HF (Hodie)-P	0.11 (-4)
H (Hodie)-P	0.03 (-4)
HF (Hodie)-Q	0.73 (-4)
H (Hodie)-Q	0.31 (-4)

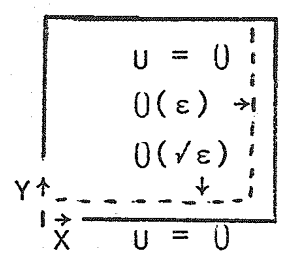
A Model Convection-Diffusion Equation

$$-\varepsilon \Delta U + U_X = 0$$

- Nonsymmetric differential operator

- Boundary layers on side-walls

$U = \sin \pi Y$



$U = 2 \sin \pi Y$

- Exact solution:

$$U(X,Y) = \frac{\exp\left(\frac{X-1}{2\varepsilon}\right)}{\sin \tau}\left[2 \sinh \tau X - \exp\left(\frac{1}{2\varepsilon}\right) \sinh \tau(X-1)\right] \sin \pi Y$$

$$\tau = \sqrt{1 + 4\pi^2\varepsilon^2} \Big/ 2\varepsilon$$

Difference Equations for the Model Convection-Diffusion Equation

- Exponentially-Weighted Second-Order

$$L_2^H\, U^H \equiv \begin{bmatrix} & 1 & \\ e^{H/2\varepsilon} & -\left(4 + \frac{H^2}{4\varepsilon^2}\right) & e^{-H/2\varepsilon} \\ & 1 & \end{bmatrix} U = 0$$

- Central-Difference Second-Order

$$L_2^H\, U^H \equiv \begin{bmatrix} & 1 & \\ \left(1 + \frac{H}{2\varepsilon}\right) & -4 & \left(1 - \frac{H}{2\varepsilon}\right) \\ & 1 & \end{bmatrix} U = 0$$

Difference Equations for the Model Convection-Diffusion Equation

- Hodie Method:

$$L_4^H U^H \equiv \begin{bmatrix} \left(1 + \frac{H}{2\varepsilon}\right) & 4 & \left(1 - \frac{H}{2\varepsilon}\right) \\ \left(4 + \frac{2H}{\varepsilon} + \frac{H^2}{2\varepsilon^2}\right) & -\left(20 + \frac{H^2}{\varepsilon^2}\right) & \left(4 - \frac{2H}{\varepsilon} + \frac{H^2}{\varepsilon^2}\right) \\ \left(1 + \frac{H}{2\varepsilon}\right) & 4 & \left(1 - \frac{H}{2\varepsilon}\right) \end{bmatrix} U = 0$$

- Compact Method:

$$L_4^H U^H \equiv \begin{bmatrix} \left\{1 + \left(1 - \frac{H^2}{48\varepsilon^2}\right) E^{H/2\varepsilon}\right\} \Big/ 2 & \left\{4 - \frac{5}{48}\frac{H^2}{\varepsilon^2}\right\} & \left\{1 + \left(1 - \frac{H^2}{48\varepsilon^2}\right) E^{-H/2\varepsilon}\right\} \Big/ 2 \\ \left\{-1 + 5\left(1 - \frac{H^2}{48\varepsilon^2}\right) E^{H/2\varepsilon}\right\} & -\left\{20 + \frac{25}{24}\frac{H^2}{\varepsilon^2}\right\} & \left\{-1 + 5\left(1 - \frac{H^2}{48\varepsilon^2}\right) E^{-H/2\varepsilon}\right\} \\ \left\{1 + \left(1 - \frac{H^2}{48\varepsilon^2}\right) E^{H/2\varepsilon}\right\} \Big/ 2 & \left\{4 - \frac{5}{48}\frac{H^2}{\varepsilon^2}\right\} & \left\{1 + \left(1 - \frac{H^2}{48\varepsilon^2}\right) E^{-H/2\varepsilon}\right\} \Big/ 2 \end{bmatrix} U = 0$$

Numerical Experiments on the Model Convection-Diffusion Equation

Finest mesh = 33 x 33

No. of grids = 4

Fixed cycling MG - algorithm

Relative Error in L_2-Norm

Method	Error
Second-order CD	Diverged
Exponentially weighted second-order	1.64 (-2)
Iterative Improvement-c	1.43 (-3)
HF (Hodie)-P	Diverged
H (Hodie)-P	1.29 (-3)
HF (compact)	1.13 (-3)
H (compact)	1.56 (-4)

2-D Navier-Stokes Equations in Stream Function (ψ) / Vorticity (ω) Form

- $\nabla^2 \psi = -\omega$
- $\nabla^2 \omega + Re\,(\psi_x \omega_y - \psi_y \omega_x) = 0$

Allen and Southwell Formulation for the Vorticity Equation

Write the vorticity equation as

- $\omega_{xx} - \mathrm{Re}\, \psi_y\, \omega_x = r\,(x, y);$ (1)
- $\omega_{yy} + \mathrm{Re}\, \psi_x\, \omega_y = -r\,(x, y)$ (2)

Transform (1) locally for $x_p - h \leqslant x \leqslant x_p + h,\ y = y_p$ by

- $\omega = \zeta \exp \{-f\,(x, y_p)\}$

  where $f\,(x, y_p) = -\frac{1}{2}\mathrm{Re}\int_{x_p}^{x} \psi_y\,(z, y_p)\, dz$

Transform (2) locally for $x = x_p,\ y_p - h \leqslant y \leqslant y_p + h$ by

- $\omega = \eta \exp \{-g\,(x_p, y)\}$

  where $g\,(x_p, y) = \frac{1}{2}\mathrm{Re}\int_{y_p}^{y} \psi_x\,(x_p, z)\, dz$

Equations for Transformed Variables ζ, η

Equation for ζ:

- $$\zeta_{xx} + \left(\frac{1}{2} \mathrm{Re}\, \psi_{xy} - \frac{1}{4} \mathrm{Re}^2\, \psi_y{}^2\right) \zeta = r\,(x, y_P)\, \exp\left\{f\,(x, y_P)\right\} \quad (3)$$

Equation for η:

- $$\eta_{yy} + \left(-\frac{1}{2} \mathrm{Re}\, \psi_{xy} - \frac{1}{4} \mathrm{Re}^2\, \psi_x{}^2\right) \eta = -\, r\,(x_P, y)\, \exp\left\{g\,(x_P, y)\right\} \quad (4)$$

Add (3) and (4) at the grid point (x_P, y_P):

- $$(\zeta_{xx})_P + (\eta_{yy})_P = \frac{1}{4} \mathrm{Re}^2 \left[(\psi_x)_P^2 + (\psi_y)_P^2\right] \omega_P$$

Compact Difference Equations

$$\begin{bmatrix} 1 & 4 & 1 \\ 4 & -20 & 4 \\ 1 & 4 & 1 \end{bmatrix} \psi = -\frac{h^2}{24} \begin{bmatrix} 1 & 10 & 1 \\ 10 & 100 & 10 \\ 1 & 10 & 1 \end{bmatrix} \omega$$

$$\begin{bmatrix} -(e^f + e^g) + \frac{Re^2 h^2}{48}\left(\psi_x^2 + \psi_y^2\right) & 2 + 10\left[-e^g + \frac{Re^2 h^2}{48}\left(\psi_x^2 + \psi_y^2\right)\right] & -(e^f + e^g) + \frac{Re^2 h^2}{48}\left(\psi_y^2 + \psi_x^2\right) \\ 2 + 10\left[-e^f + \frac{Re^2 h^2}{48}\left(\psi_x^2 + \psi_y^2\right)\right] & 40 + \frac{100 Re^2 h^2}{48}\left(\psi_x^2 + \psi_y^2\right) & 2 + 10\left[-e^f + \frac{Re^2 h^2}{48}\left(\psi_x^2 + \psi_y^2\right)\right] \\ -(e^f + e^g) + \frac{Re^2 h^2}{48}\left(\psi_x^2 + \psi_y^2\right) & 2 + 10\left[-e^g + \frac{Re^2 h^2}{48}\left(\psi_x^2 + \psi_y^2\right)\right] & -(e^f + e^g) + \frac{Re^2 h^2}{48}\left(\psi_x^2 + \psi_y^2\right) \end{bmatrix} \omega = 0$$

Hodie Difference Equations

$$\begin{bmatrix} 1 & 4 & 1 \\ 4 & -20 & 4 \\ 1 & 4 & 1 \end{bmatrix} \phi = -\frac{H^2}{2} \begin{bmatrix} & 1 & \\ 1 & 8 & 1 \\ & 1 & \end{bmatrix} \omega$$

$$\begin{bmatrix} \alpha_6 & \alpha_2 & \alpha_5 \\ \alpha_3 & \alpha_0 & \alpha_1 \\ \alpha_7 & \alpha_4 & \alpha_8 \end{bmatrix} \omega = 0$$

$$\alpha_0 = -120 + \text{Re}^2\, H^2\, (u_0^2 + v_0^2) - \text{Re}\, H(u_1 - u_3) - \text{Re}\, H(v_2 - v_4)]$$

$$\alpha_1 = 4 - \frac{\text{Re}\ H}{4} (4\, u_0 + 3\, u_1 + u_2 - u_3 + u_4)$$

$$+ \frac{\text{Re}^2 H^2}{8} [4\, u_0^2 + u_0(u_1 - u_3) + v_0(u_2 - u_4)]$$

$$\alpha_2 = 4 - \frac{\text{Re}\ H}{4} (4v_0 + v_1 + 3v_2 + v_3 - v_4)$$

$$+ \frac{\text{Re}^2 H^2}{8} [4v_0^2 + u_0(v_1 - v_3) + v_0\ (v_2 - v_4)]$$

$$\alpha_3 = 4 + \frac{\text{Re}\ H}{4} (4u_0 - u_1 + u_2 + 3u_3 + u_4)$$

$$+ \frac{\text{Re}^2 H^2}{8} [4u_0^2 - u_0(u_1 - u_3) - v_0(u_2 - u_4)]$$

$$\alpha_4 = 4 + \frac{R_E H}{4} [4v_0 + v_1 - v_2 + v_3 + 3v_4)$$

$$+ \frac{R_E^2 H^2}{8} [4v_0^2 - u_0(v_1 - v_3) - v_0(v_2 - v_4)]$$

$$\alpha_5 = 1 - \frac{R_E H}{2} (u_0 + v_0) + \eta$$

$$\alpha_6 = 1 + \frac{R_E H}{2} (u_0 - v_0) - \eta$$

$$\alpha_7 = 1 + \frac{R_E H}{2} (u_0 + v_0) + \eta$$

$$\alpha_8 = 1 - \frac{R_E H}{2} (u_0 - v_0) - \eta$$

$$\eta = \frac{R_E H}{8} (v_1 - v_3 + u_2 - u_4) + \frac{R_E^2 H^2}{4} u_0 v_0$$

$U = \phi_Y$ AND $V = - \phi_X$

NUMERICAL EXAMPLES

$$\Omega = (0,1) \times (0,\ 1) \quad ,\ 0 \le Re \le 100$$

(A) $\phi = -\ e^{x-y}$ AND $\omega = 2\ e^{x-y}$ ON $\partial\Omega$.

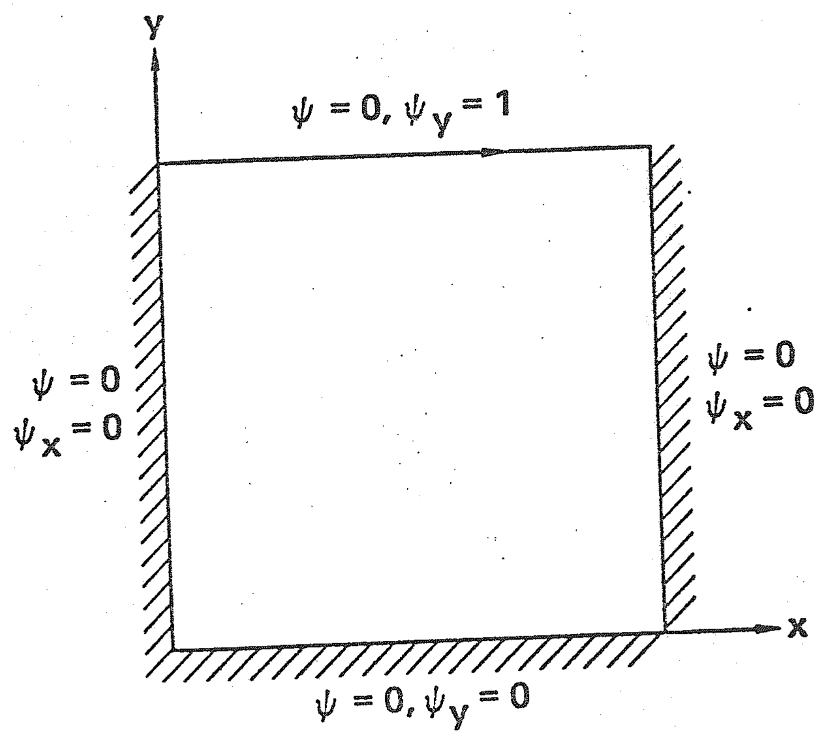
(B) $\phi = y - \frac{1}{\pi} \sin 2\pi y\ e^{-\alpha x}$ AND

$$\omega = (\alpha^2 - 4\pi^2) \frac{1}{\pi} \sin 2\pi y\ e^{-\alpha x} \text{ ON } \partial\Omega,$$

WHERE

$$\alpha = \left(-\ Re \pm \sqrt{Re^2 + 16\ \pi^2}\right) \Big/ 2$$

Flow in a Driven-Square Cavity



Boundary Conditions:

- Use Pade' relation for a function and its derivatives
- Vorticity at the moving wall is given by:

$$\omega\,(i, jmax) = -\left[\frac{12}{h^2}\,\psi\,(i, jmax-1) + \frac{6}{h}\left\{1 + \psi_y\,(i, jmax-1)\right\} + \psi_{yy}\,(i, jmax-1)\right]$$

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Asymptotic Convergence Factor μ for the 2-D Steady Navier-Stokes Equations

$\Omega = (0,1) \times (0, 1)$, Number of Grids = 4, and Finest Mesh = 65 x 65 with Spacing $h = 0.015625$

Relaxation Scheme	Case (A)		Case (B)		Case (C)	
	Re = 0	Re = 100	Re = 0	Re = 100	Re = 0	Re = 100
5-point Gauss-Seidel	0.90	0.92	0.78	0.83	0.92	0.95
9-point compact Gauss-Seidel	0.85	0.87	0.72	0.76	0.86	0.87

Numerical Experiments on the Steady Navier-Stokes Equations

Flow in a Driven-Cavity

Finest mesh = 65 x 65, No. of grids = 4

$h = 1/64$

Fixed cycling MG - algorithm

Relative Error in L_2-Norm ($R_E = 100$)

Method	Error
Exponentially weighted second-order	1.76 (+2)
H (compact)	1.39 (+1)
H (Hodie)-P	1.10 (+1)

Unigrid Method – Basic Outline

- Consider the relaxation scheme described in terms of a directional iteration operator

$$G^h(U^h, d^h) = U^h - \omega \frac{< L^h U^h - f^h, d^h >}{< L^h d^h, d^h >} d^h ,$$

where $d^h = e^h_k$ (k^{th} coordinate vector) , $k = 1, \ldots n$ for the Gauss-Seidel relaxation.

- Consider a set of grids defined by $h_q = 2^q h$, $0 \leq q \leq m$.

- Define direction sets on h_q as

$$D^h_q = \left(d^{h_q}_1, d^{h_q}_2, \ldots d^{h_q}_n\right), 0 \leq q \leq m , \quad \text{where} \quad d^{h_q}_k = I^h_{h_q} d^{h_q}_k .$$

- The directions on level q are just the relaxation directions on grid h_q transferred to grid $h = h_o$.
- One unigrid cycle on level q consists first of ν relaxation sweeps with directions $d_k^{h_q}$, $k = 1, \ldots n$, followed for $q < m$ by μ cycles on level $q + 1$ and for $q = m$ by ν_c more sweeps.
- For a rectangular domain,

$$d_{k,\ell}^{h_q}(i, j) = \begin{cases} (2^q - |k - i|)\,(2^q - |\ell - j|) & \text{for } |k - i|, |\ell - j| \leq 2^q \\ 0 & \text{otherwise.} \end{cases}$$

- In unigrid, every coarse-grid computation is immediately reflected in the fine-grid approximation, resulting in a version of multigrid described solely in terms of fine-grid computations.

- Significantly more arithmetic work and less efficient than multigrid.
- Requires less storage and results in a short code.
- Equivalent to multigrid under the variational conditions

$$L^{2h} = I_h^{2h} L^h I_{2h}^h$$

$$I_h^{2h} = \left(I_{2h}^h\right)^T .$$

- Can serve as a multigrid software simulator.

Unigrid for Non-Symmetric Operators

$$G^H (U^H, D^H) = U^H - \omega \frac{\langle L^H U^H - F^H, L^H D^H \rangle}{\langle L^H D^H, L^H D^H \rangle} D^H$$