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Multigrid Applied to Two-Dimensional Vesicle Suspensions

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We are interested in simulating high concentration suspensions of deformable capsules suspended in a Stokesian fluid. Such capsules are typically modeled as infinitesimally thin membranes filled with a viscous fluid. In particular, we are interested in a specific type of capsule known as a *vesicle*. Vesicles are closed inextensible lipid membranes containing a viscous fluid. The membrane mechanics are characterized by bending resistance (derived by a Helfrich energy, or a minimization of the L^2 norm of the mean curvature), zero resistance to shear, and inextensibility (infinite resistance to elongation). Although we focus on vesicles, the resulting methodologies should be applicable in a broad range of phenomena in complex fluids and suspensions.

Vesicle evolution dynamics are given by a quasi-static Stokes equation driven by interface jump conditions derived by a balance of forces on the membrane. Given the position of the vesicles, the jump conditions can be evaluated, a Stokes problem with interface conditions can be solved, and the velocity at the interface advances the vesicle to its new location. We use a boundary integral formulation that results in a system of non-linear integro-differential equations for the vesicles's position and an auxiliary field that is a "tension" (it acts as a Lagrange multiplier to enforce the inextensibility constraint). Below we summarize the main equations.

Consider a single vesicle $\gamma \in \mathbb{R}^2$ parameterized by the periodic function $\mathbf{x}$. The equations governing the motion of the vesicle are

$$\frac{d\mathbf{x}}{dt} = \mathcal{S}[-\mathbf{x}_{ssss} + (\sigma \mathbf{x}_s)_s], \qquad \mathbf{x}_s \cdot \frac{d\mathbf{x}_s}{dt} = 0,$$

where σ is the tension and $\mathcal{S}[\mathbf{f}](\mathbf{x}) = \int_\gamma G(\mathbf{x}-\mathbf{y})\mathbf{f}(\mathbf{y})ds_{\mathbf{y}}$, where $G(\mathbf{x}) = \frac{1}{4\pi\mu}\left(-\log|\mathbf{x}| + \frac{\mathbf{x}\otimes\mathbf{x}}{|\mathbf{x}|^2}\right)$, is the single-layer potential for Stokes flow. We introduce the operators $\mathcal{B}$ (bending), $\mathcal{T}$ (tension), and $\mathcal{D}$ (divergence) so that the governing equations are

$$\frac{d\mathbf{x}}{dt} = \mathcal{S}[-\mathcal{B}\mathbf{x} + \mathcal{T}\sigma], \qquad \mathcal{D}\frac{d\mathbf{x}}{dt} = 0.$$

Discretizing in time and linearizing, the vesicle position and tension are updated by solving

$$\begin{bmatrix} I + \Delta t\mathcal{S}\mathcal{B} & -\Delta t\mathcal{S}\mathcal{T} \\ -\mathcal{D}\mathcal{S}\mathcal{B} & \mathcal{D}\mathcal{S}\mathcal{T} \end{bmatrix} \begin{bmatrix} \mathbf{x}^{N+1} \\ \sigma^{N+1} \end{bmatrix} = \begin{bmatrix} \mathbf{x}^{N} \\ 0 \end{bmatrix}. \tag{1}$$

This system is solved with a matrix-free fast-multipole accelerated GMRES for the new position and tension. However, the number of iterations depends on N since the system is ill-conditioned. This motivates the use of multigrid. In this talk we present the overall scheme and preliminary results that demonstrate the effectiveness of the proposed scheme.

As a first step, we are looking for efficient solvers for the Schur complement of the tension, $\mathcal{L} := \mathcal{D}\mathcal{S}\mathcal{T}$. $\mathcal{L}$ must be inverted for explicit schemes or it can be used to form block preconditioners for implicit schemes when solving (1). Standard smoothers applied to integral operators such as $\mathcal{L}$ generally reduce low frequencies in the error. We propose using GMRES applied to the high frequency components as a smoother. In more detail, the smoother for $\mathcal{L}\sigma = b$ is $\sigma^{N+1} = P\sigma^N + \sigma_H$, where σ_H solves $(I-P)\mathcal{L}\sigma_H = (I-P)b - (I-P)\mathcal{L}P\sigma^N$, and P is the projection onto the low frequency space.

We use a V-cycle multigrid solver for $\mathcal{L}$ and report results in Table 1. We see that only a few GMRES iterations are required to apply the smoother. We set the GMRES tolerance of the smoother to 10^{-8}, the coarsest level to $N = 8$, and stop the solver when the relative error is less than 10^{-8}. We are currently experimenting with using multigrid to precondition a GMRES solver for $\mathcal{L}$.

N	ρ	nV	GMRES
16	0.138	10	2
32	0.365	19	4
64	0.448	23	8
128	0.573	35	13

Table 1: N is the problem size, ρ is the average convergence rate, nV is the number of V-cycles, and GMRES is the number of GMRES iterations required by the smoother at the finest level.