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**HyGA: A Hybrid Geometric+Algebraic Multigrid Solver
for Weighted-Residual Methods with Hierarchical Meshes**

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Multigrid methods are efficient and effective for solving large-scale sparse linear systems, especially those from the discretizations of partial differential equations (PDEs). Generally speaking, multigrid may be classified into *geometric* (GMG) and *algebraic* (AMG), each of which has advantages and disadvantages. The primary advantages of GMG include its better convergence with smooth solutions, better efficiency in terms of computational time, and also better efficiency in storage especially with a matrix-free implementation. However, the GMG tends to have difficulties for non-smooth problems and irregular domains, and it is difficult to implement and to scale with unstructured meshes, especially for domains with complex topologies. The primary advantages of AMG include its simplicity and flexibility, and its robustness for non-smooth solutions. However, AMG tends to be more expensive, especially for its setup stage. It also tends to generate denser matrices than GMG and cannot be easily implemented in a matrix-free fashion, so it has higher memory and computational costs. A more serious problem is that AMG has no guarantee on the condition numbers of the matrices at the coarse level, which may lead to non-convergence. Table 1 summarizes a comparison of geometric and algebraic multigrid methods.

Table 1: Advantages and disadvantages of GMG and AMG. The goal of HyGA is to combine their advantages.

	geometric MG	algebraic MG	HyGA
set up stage	inexpensive	expensive	inexpensive
memory requirement	low	high	low
matrix free	possible	impossible	possible at finer levels
computational cost	low	high	low
matrix quality	high	no control	high at finer levels
convergence	faster	slower	faster
implementation	difficult	simple	intermediate
irregular domains	not robust	robust	robust

The goal of this work is to develop a novel and general *hybrid geometric+algebraic*

multigrid method, or *HyGA*, which will combine the advantages and overcome the disadvantages of the GMG and AMG shown in Table 1. We focus on linear systems arising from a weighted residual method for linear PDEs over unstructured meshes. Linear PDEs cover a wide range of mathematical models, such as Poisson equations, and the weighted residuals are general discretization techniques, which can unify many commonly used finite elements and finite differences by defining different weighting functions. In addition, unstructured meshes are very general in resolving complex geometries. An important contribution of this work is a new, unified derivation of restriction and prolongation operators for weighted residuals with hierarchical basis functions. This rigorous analysis coupled with the generality of our formulation will allow us to address a large class of linear systems arising from scientific and engineering applications.

At the algorithmic level, the main contribution of this work is the hybrid multigrid solver *HyGA*, which combines a high-quality hierarchical mesh generator, a geometric multigrid solver with a multilevel weighted residual formulation, and an algebraic multigrid solver at the coarsest levels. Our proposed algorithm may be summarized as follows. Our hierarchical mesh generator starts from a good-quality coarse unstructured mesh that is sufficiently accurate for representing the topology and for reconstructing the geometry of the domain. It iteratively refines the mesh with guaranteed mesh quality (by uniform refinements) and geometric accuracy (by high-order boundary reconstruction). Because this refinement involves only local operations, it is efficient and scalable. We apply GMG with a multilevel weighted-residual formulation on these hierarchical meshes using our unified prolongation and restriction operators, which are coupled with PDE rediscretizations over coarse meshes. This combination allows efficient matrix-free implementations on the coarse levels, and also effective control of the sparsity and the condition numbers of the resulting matrices. At the coarsest level, we employ the classical AMG, which allow us to resolve complex topologies easily and handle non-smooth solutions robustly. In addition, the relatively high setup cost and memory requirement of AMG become negligible due to the significantly reduced problem size. Therefore, our approach effectively combines the rigor, high accuracy and runtime-and-memory efficiency of GMG with the flexibility of AMG, and at the same time it is relatively easy to implement.

We present numerical experiments to demonstrate the effectiveness of *HyGA* compared with GMG and AMG for 2-D and 3-D problems. Our experimental results show that both *HyGA* and GMG deliver better convergence and better efficiency than AMG for weighted residual methods over unstructured meshes. In addition, *HyGA* delivers similar, and sometimes better efficiency and convergence rate than GMG due to the flexibility of AMG at the coarse levels. In summary, our theoretical analysis and numerical experimentations demonstrate that *HyGA* is an effective and robust multigrid approach for weighted-residual methods with hierarchical meshes.