A PRECONDITIONER ON HIGH-ORDER FINITE ELEMENT METHODS*

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Even if the high-order finite element method has many advantages for solving a uniformly self adjoint elliptic operator such as

$$Lu := -\nabla \cdot \mathbf{A}\nabla u + c_0 u \quad \text{in} \quad \Omega = [-1,1]\times[-1,1]$$

with boundary conditions ($\Gamma_L = \Gamma_D(L) \cup \Gamma_N(L)$)

$$u = 0 \quad \text{on} \quad \Gamma_D(L), \quad \mathbf{n}\cdot\mathbf{A}\nabla u = 0 \quad \text{on} \quad \Gamma_N(L),$$

one may have a difficulty controlling condition numbers occurred from spectral element discretizations which makes it uneasy to use iterative methods. In order to alleviate such a situation, we take a lower order finite element preconditioner operator corresponding to

$$Bv := -\nabla \cdot \nabla u + b_0 u \quad \text{in} \quad \Omega$$

with boundary conditions ($\Gamma_B = \Gamma_D(B) \cup \Gamma_N(B)$)

$$v = 0 \quad \text{on} \quad \Gamma_D(B), \quad \mathbf{n}\cdot\nabla v = 0 \quad \text{on} \quad \Gamma_N(B).$$

Let $\{\eta_k\}_{k=0}^N$ be the standard Legendre-Gauss-Lobatto (=:LGL) points in $[-1,1]$. By translations from I to a j^{th} subinterval $I_j := [x_{j-1}, x_j]$ we denote $\{\xi_k^j\}_{k=0}^N$ as the $k^{th}-$LGL points in each subinterval I_j for $j = 1, 2, \cdots, M$. Let $\mathcal{P}_N^h$ be the subspace of $C[-1,1]$ which consists of piecewise polynomials with support $I_j = [x_{j-1}, x_j]$ whose degree is less than or equal to N. For the space $\mathcal{P}_N^h$, we choose a *piecewise Lagrange polynomial basis functions* denoted as $\{\phi_k^j(x)\}$ supported in I_j for $j = 1, \cdots, M$. Let $\mathcal{V}_N^h$ be the space of all *piecewise Lagrange linear functions* $\psi_k^i(x)$. Define an interpolation operator $\mathcal{I}_N^h : C[-1,1] \to \mathcal{P}_N^h(I)$ such that

$$(\mathcal{I}_N^h v)(\xi_\mu) = v(\xi_\mu), \quad v \in C[-1,1].$$

First, we set up the following relations for $v \in \mathcal{V}_N^h$

$$c\|v\| \le \|\mathcal{I}_N^h v\| \le C\|v\|, \quad c\|v\|_1 \le \|\mathcal{I}_N^h v\|_1 \le C\|v\|_1,$$

where two positive constants c and C do not independent of the mesh size $h_j = x_j - x_{j-1}$ and the degree N of piecewise polynomial. Let $(\hat{\mathrm{L}}_N^h)$ and $\hat{\mathbf{B}}_N^h$ be finite element stiffness matrices corresponding to L and B respectively . Then we will show the preconditioned system

$$(\hat{\mathbf{B}}_N^h)^{-1}\hat{L}_N^h$$

has positive eigenvalues which are independent of the mesh size $h_j = x_j - x_{j-1}$ and the degree N of piecewise polynomial.

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