Consider the saddle point problem
\[
A \begin{bmatrix} u \\ x \end{bmatrix} = \begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} u \\ x \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}.
\]
Assuming that \(A\) is s.p.d., this problem is transformed, by a (computable) projection \(\pi\) such that \(B\pi = \hat{B}\), to an equivalent s.p.d. problem for \(u\),
\[
\left(I - \pi^T \right) A \left(I - \pi \right) + \pi^T A \pi \right) u = \left(I - \pi^T \right) f.
\]
We present a set of conditions for a smoother \(M^{-1}\) and an interpolation matrix \(\Pi\), such that if a current iterate in the resulting two–grid method belongs to the subspace \(\text{Null}(B)\), then after smoothing the iterate stays in \(\text{Null}(B)\), and finally, the (interpolated) coarse–grid correction also stays in the subspace \(\text{Null}(B)\).

Thus a multigrid method can be devised without explicit knowledge of a computable basis of \(\text{Null}(B)\). The tools needed are: computable projections \(\pi_k\), such that \(\pi_k^T\) are also computable, interpolation matrices \(P_k\) for the \(u\)–variable and interpolation matrices \(Q_k\) for the second unknown \(x\), at all levels \(k \geq 0\). Let \(B_0 = B\) and \(A_0 = A\) (i.e., \(k = 0\) stands for the finest level). The projections \(\pi_k\) have the form \(R_k B_k\) and satisfy \(B_k \pi_k = B_k\). There is a common “null–space preserving” assumption on \(Q_k, P_k,\) and \(B_{k+1} \equiv Q_k^T B_k P_k, B_{k+1} v_c = 0\) implies \(B_k P_k v_c = 0\).

Define a standard multigrid method based on the s.p.d. matrices
\[
(I - \pi_k^T), A_k (I - \pi_k) + \pi_k^T A_k \pi_k, \quad \lambda_k = P_{k-1} A_{k-1} P_{k-1}, \quad A_0 = A,
\]
smoothers (for given s.p.d. matrices \(M^{-1}\)), \(M^{-1}_k = (I - \pi_k) M^{-1}_k (I - \pi_k^T) + \pi_k M^{-1}_k \pi_k^T\), and (modified) interpolation matrices \(\Pi_k = P_k (I - \pi_{k+1})\). The resulting multigrid method (with zero initial iterate) keeps all iterates in \(\text{Null}(B)\) since the initial residual is \((I - \pi^T)f\) and all successive residuals also have the form \((I - \pi^T)r\).
We provide a specific construction of the (computable) projections $\pi$ as well as alternative choices of the null–space preserving smoothers $M^{-1}$ for some mixed finite element saddle–point matrices $A$.

This work was performed under the auspices of the U. S. Department of Energy by University of California Lawrence Livermore National Laboratory under contract W-7405-Eng-48.