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Nullspace preserving multigrid for saddle–point problems

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Consider the saddle point problem

$$A \begin{bmatrix} \mathbf{u} \\ \mathbf{x} \end{bmatrix} \equiv \begin{bmatrix} \mathcal{A} & \mathcal{B}^T \\ \mathcal{B} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{x} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ 0 \end{bmatrix}.$$

Assuming that $\mathcal{A}$ is s.p.d., this problem is transformed, by a (computable) projection π ($\pi^2 = \pi$) such that $\mathcal{B}\pi = \mathcal{B}$, to an equivalent s.p.d. problem for $\mathbf{u}$,

$$\left[(I - \pi^T)\mathcal{A}(I - \pi) + \pi^T \mathcal{A}\pi\right] \mathbf{u} = (I - \pi^T)\mathbf{f}.$$

We present a set of conditions for a smoother $\mathcal{M}^{-1}$ and an interpolation matrix Π, such that if a current iterate in the resulting two–grid method belongs to the subspace $\mathrm{Null}(\mathcal{B})$, then after smoothing the iterate stays in $\mathrm{Null}(\mathcal{B})$, and finally, the (interpolated) coarse–grid correction also stays in the subspace $\mathrm{Null}(\mathcal{B})$. Thus a multigrid method can be devised without explicit knowledge of a computable basis of $\mathrm{Null}(\mathcal{B})$. The tools needed are: computable projections π_k, such that π_k^T are also computable, interpolation matrices $\mathcal{P}_k$ for the $\mathbf{u}$–variable and interpolation matrices $\mathcal{Q}_k$ for the second unknown $\mathbf{x}$, at all levels $k \geq 0$. Let $\mathcal{B}_0 = \mathcal{B}$ and $\mathcal{A}_0 = \mathcal{A}$ (i.e., $k = 0$ stands for the finest level). The projections π_k have the form $\mathcal{R}_k\mathcal{B}_k$ and satisfy $\mathcal{B}_k\pi_k = \mathcal{B}_k$. There is a common "*null–space preserving*" assumption on $\mathcal{Q}_k$, $\mathcal{P}_k$, and $\mathcal{B}_{k+1} \equiv \mathcal{Q}_k^T\mathcal{B}_k\mathcal{P}_k$, $\mathcal{B}_{k+1}\mathbf{v}_c = 0$ implies $\mathcal{B}_k\mathcal{P}_k\mathbf{v}_c = 0$.

Define a standard multigrid method based on the s.p.d. matrices

$$(I - \pi_k^T)\mathcal{A}_k(I - \pi_k) + \pi_k^T\mathcal{A}_k\pi_k, \quad \mathcal{A}_k = \mathcal{P}_{k-1}^T\mathcal{A}_{k-1}\mathcal{P}_{k-1},\ \mathcal{A}_0 = \mathcal{A},$$

smoothers (for given s.p.d. matrices $\overline{\mathcal{M}}_k^{-1}$),$\mathcal{M}_k^{-1} = (I - \pi_k)\overline{\mathcal{M}}_k^{-1}(I - \pi_k^T) + \pi_k\overline{\mathcal{M}}_k^{-1}\pi_k^T$, and (modified) interpolation matrices $\Pi_k = \mathcal{P}_k(I - \pi_{k+1})$. The resulting multigrid method (with zero initial iterate) keeps all iterates in $\mathrm{Null}(\mathcal{B})$ since the initial residual is $(I - \pi^T)\mathbf{f}$ and all successive residuals also have the form $(I - \pi^T)\mathbf{r}$.

We provide a specific construction of the (computable) projections π as well as alternative choices of the null–space preserving smoothers $\mathcal{M}^{-1}$ for some mixed finite element saddle–point matrices A.

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