Suely Oliveira

Algebraic Multigrid (AMG) for saddle point systems from meshfree discretizations

The University of Iowa
Department of Computer Science
14 MLH
Iowa City
IA 52242
oliveira@cs.uiowa.edu

Meshfree discretizations construct approximate solutions to partial differential equation based on particles, not on meshes, so that it is well suited to solve the problems on irregular domains. Since the nodal basis property is not satisfied in meshfree discretizations, it is difficult to handle essential boundary conditions. In this paper, we employ the Lagrange multiplier approach to solve this problem, but this will result in an indefinite linear system of a saddle point type. We adapt a variation of the smoothed aggregation AMG method of Vaněk, Mandel & Brezina to this saddle point system. We give numerical results showing that this method is practical and competitive with other methods with convergence rates that are $\sim c/\log N$.

Meshfree discretizations are Galerkin approximations to the weak form of partial differential equations, where each unknown corresponds to a "particle" — a smooth function with compact support. In most meshfree methods these particle functions are not arbitrary, but are constructed to satisfy certain properties in order to achieve good approximation properties. In particular, we work with Reproducing Kernel Particle Methods (RKPM's). In RKPM's, an initial collection of smooth functions with compact support (kernel functions Φ_i with associated nodes x_i) are processed to construct new smooth functions with compact support (basis functions Ψ_i) which satisfy the discrete reproducing condition

$$\sum_i x_i^\alpha \, \Psi_i(x) = x^\alpha$$

where α is any multiindex $|\alpha| \leq p$.

Meshfree discretizations are Galerkin approximations to the weak form of partial differential equations, where each unknown corresponds to a "particle" — a smooth function with compact support. In most meshfree methods these particle functions are not arbitrary, but are constructed to satisfy certain properties in order to achieve good approximation properties. In particular, we work with Reproducing Kernel Particle Methods (RKPM's). In RKPM's, an initial collection of smooth functions with compact support (kernel functions Φ_i with asso-

ciated nodes x_i) are processed to construct new smooth functions with compact support (basis functions Ψ_i) which satisfy the discrete reproducing condition

$$\sum_i x_i^\alpha \, \Psi_i(x) = x^\alpha$$

where α is any multiindex $|\alpha| \leq p$.

Since there is no particular relationship between the nodes and the kernel functions, the nodal basis property fails for the RKPM and other standard meshfree methods. The nodal basis property can be re-established by modifying the basis functions using a variety of techniques (singular kernel functions, hybrid finite element/meshfree methods, etc.). Instead we consider a Lagrange multiplier method. Lagrange multiplier methods are often vulnerable to numerical instabilities if the discrete Babuska–Brezzi conditions fail, as they typically do for meshfree methods. To overcome this problem, we construct a separate family of basis functions for the boundary for approximating the Lagrange multiplier function.

The starting point for our preconditioner is a smoothed aggregation method of Vaněk, Mandel & Brezina. However, to construct our smoothed interpolation operator we use a different way of deriving the interpolation operator to that of Vaněk *et al.*

We have a large-scale saddle point system to solve. Since we need to maintain the discrete Babuska–Brezzi condition for the grid coarsenings, we also coarsen the boundary discretization. Smoothed aggregation techniques similar to those for the interior of the domain, are used to construct the boundary interpolation and restriction operators. The coarse grid operators are constructed using a Galerkin approach — pre-multiplying the fine-grid saddle-point operator with the block diagonal matrix consisting of the interpolation operators for the interior and the boundary, and post-multiplying by the transpose of the above block-diagonal matrix. We use a saddle-point JOR-type method for the pre- and post-smoother. The resulting multigrid V-cycle operator is then used to precondition GMRES.

The numerical results we obtain are certainly encouraging, giving a number of iterations which grows like $c \log N \log(1/\epsilon)$ where N is the number of unknowns and ϵ is the error tolerance. Unfortunately, the number of flops per V-cycle is not constant because of reductions in the sparsity of the course-grid operators, and the smoother used. In spite of these problems, the method out-performed the methods we compared it with (including straight aggregation, using the smoother as a preconditioner, and no preconditioner at all), especially for larger problems. Good performance can be seen both in the number of iterations and the total time taken. This is joint work with Koung Hee Leem and David Stewart.