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**Multilevel Preconditioners for Nonselfadjoint or Indefinite
Orthogonal Spline Collocation Problems**

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We develop and study symmetric multilevel preconditioners for the computation of the orthogonal spline collocation (OSC) solution of a Dirichlet boundary value problem (BVP) with a nonselfadjoint or an indefinite operator. The OSC solution is sought in the space of piecewise Hermite bicubic spline functions defined on a uniform partition. We consider an additive and a multiplicative multilevel preconditioners that are used with the preconditioned conjugate gradient (PCG) method. Results and algorithms presented in this paper are closely related to those in [1], [2], [3], and [4].

Let Ω be a unit square $(0, 1) \times (0, 1)$ with the boundary $\partial\Omega$, and let $x = (x_1, x_2)$. We consider a BVP

$$Lu \equiv \sum_{i,j=1}^2 a_{ij}(x)u_{x_i x_j} + \sum_{i=1}^2 b_i(x)u_{x_i} + c(x)u = f(x), \quad x \in \Omega, \quad u = 0 \quad \text{on } \partial\Omega. \quad (1)$$

Operator L could be non-selfadjoint or indefinite in L^2 inner product. We assume that the principal part of L satisfies the uniform ellipticity condition and that BVP (1) has a unique solution in $H^2(\Omega)$.

Let π_0 be a uniform coarsest rectangular partition of Ω . We obtain a set of partitions $\{\pi_k\}_{k=0}^K$ by standard coarsening, and let $V_0 \subset V_1 \subset \dots \subset V_K \equiv V_h$ be the set of corresponding nested spaces of piecewise Hermite bicubics that vanish on $\partial\Omega$. Let \sum denote the two-dimensional composite Gauss quadrature corresponding to partition π_h with 4 nodes in each element. Let \mathcal{G}_h denote the corresponding set of Gauss points. The OSC discretization of BVP (1) is defined by

$$u_h \in V_h, \quad Lu_h(\xi) = f(\xi), \quad \xi \in \mathcal{G}_h, \quad (2)$$

and it can be written as the operator equation $L_h u_h = f_h$ in the Hilbert space V_h with the inner product $(v, w)_h = \sum vw$.

Let $\{\psi_{k,j}\}_{j=1}^{N_k}$ be the standard finite element basis of V_k consisting of products of one-dimensional value and slope basis functions. Using space decomposition

$$V_h = V_0 + \sum_{k=1}^J \sum_{j=1}^{N_k} V_{k,j}, \quad V_{k,j} = \text{span}(\psi_{k,j}),$$

we define and study multilevel additive B_a and multiplicative B_m preconditioners for solving the normal equation $L_h^* L_h u_h = L_h^* f_h$, where L_h^* is the adjoint to L_h . The implementation of B_a and B_m is based on relationships between basis functions for two consecutive partitions and the implementation of B_m is similar to that for V(1,1)-cycle with the Gauss-Seidel smoothing. A problem on the coarsest partition is assumed sufficiently small, and it is solved exactly. The computational cost of the preconditioning algorithms is $O(N_K)$. The following is our main result.

Theorem. *There are positive independent of h and K constants α_a , β_a , α_m , and β_m , such that*

$$\begin{aligned} \alpha_a (B_a v, v)_h &\leq (L_h^* L_h v, v)_h \leq \beta_a (B_a v, v)_h, & v \in V_h, \\ \alpha_m (B_m v, v)_h &\leq (L_h^* L_h v, v)_h \leq \beta_m (B_m v, v)_h, & v \in V_h. \end{aligned} \quad (3)$$

To obtain these results, we prove the key assumptions in the general theory of Schwarz methods presented in [4], and use the inequalities

$$C^{-1} \|v\|_{H^2(\Omega)}^2 \leq a_h(v, v) \leq C \|\Delta v\|_{L^2(\Omega)}^2, \quad v \in V_h,$$

obtained in [2]. We present numerical results that demonstrate the efficiency of our preconditioning algorithms.

References

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