WEIGHTED-NORM FIRST-ORDER SYSTEM LEAST SQUARES (FOSLS) FOR PROBLEMS WITH CORNER SINGULARITIES

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Abstract.
A weighted-norm least-squares method is considered for the numerical approximation of solutions that have singularities at the boundary. While many methods suffer from a global loss of accuracy due to boundary singularities, the least-squares method can be particularly sensitive to a loss of regularity. The method we describe here requires only a rough lower bound on the power of the singularity and can be applied to a wide range of elliptic equations. Optimal order discretization accuracy is acheived in weighted H^1, and functional norms and L^2 accuracy is retained for boundary vaule problems with a dominant div-curl operator. Our analysis, including interpolation bounds and several Poincaré type inequalities, are carried out in appropriately weighted Sobolev spaces. Numerical results confirm the error bounds predicted in the anaylsis.

1. Introduction. Many elliptic boundary value problems have the fortunate property of a guaranteed smooth solution as long as the data and domain are smooth. However, many problems of interest are posed in nonsmooth domains and, as a consequence, lose this property at a finite number of points on the boundary in two dimensions or along curves on the boundary in three dimensions. In this paper, we study problems that have nonsmooth solutions at *irregular boundary points*, that is, points that are corners of polygonal domains, locations of changing boundary condition type, or both.

Standard solution techniques applied to such boundary value problems suffer from a global loss of accuracy due to the reduced smoothness of the solution. Several approaches are used to combat this so-called *pollution effect*. The most common of these in practice is systematic local mesh refinement near the singularities. But even local refinement of finite element subspaces of H^1 fails to converge to a solution that is not in H^1.

If a basis for the singular functions is known, it can be incorporated directly into the finite element space. In [10, 11], this approach is shown to restore optimal convergence throughout the domain. For some two-dimensional problems, the singular basis functions are known and can be included in the finite element space. For the other problems, or in three dimensions, the exact character of the singular functions is less well understood.

Least-squares methods based on inverse norms can be effective for problems with discontinuous coefficients and data in H^{-1}. For example, in [8, 12, 17, 20–22], the functional is posed in terms of H^{-1} norms rather than L^2 norms, resulting in optimal L^2 approximations to the solution. A more recent approach, called FOSLL*, uses an inverse norm induced by the equations, and is shown in [14, 19] to be more efficient than the H^{-1} norm methods. Other methods for alleviating the pollution effect can be found in [2, 6, 7, 13, 18].

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We investigate a technique within the first-order system least-squares framework that requires only the power of the singularity (not the actual singular solution), recovers optimal order accuracy in the weighted H^1, L^2 and functional norms, and retains L^2 convergence even near the singularity. In practice, this method can be used with only a rough estimate of the power of the singularity, which can be adaptively determined if unknown. In [3] a weighted norm is used in a least-squares functional in conjunction with a sequence of graded meshes to alleviate the pollution effect. The method we use in this paper is similar, but our analysis allows for more aggressive weighting within a wider class of problems. In addition, we prove several Poincaré type inequalities in weighted Sobolev spaces under different boundary conditions that, in addition to being necessary for our main result, may be of independent interest.

The basic FOSLS approach is to recast the original system as an appropriate first-order system and apply an L^2 minimization principle over the residual of the equations. If possible, this reformulation is done by minimizing a functional whose quadratic part is equivalent to the product H^1 norm, indicating that the process is similar to solving a weakly coupled system of Poisson-like equations. This equivalence also guarantees optimal H^1 accuracy for standard discretizations. For problems with singular solutions, the L^2 based functional fails to be H^1 equivalent and, as a consequence, standard discretizations suffer from the pollution effect. The method tries to approximate a singular solution by minimizing the error in the H^1 norm (which it is not able to do) at the expense of accuracy in the entire domain. The weighted-norm least-squares method replaces the L^2 norms in the FOSLS functional with locally weighted L^2 norms, making the functional norm equivalent to a weighted H^1 norm. With an appropriate weighting function, we then achieve optimal accuracy in this weighted H^1 space and convergence in the L^2 measure.

We use Poisson's equation on a domain with a reentrant corner as a model problem and as the formal setting for analysis. The resulting div-curl system is a basic component of the FOSLS formulation of many elliptic problems. The analysis in this paper is restricted to two dimensions. However, the approach suggests a natural generalization to three dimensions.

2. Singular solutions and preliminaries. For vector function $\mathbf{u} = (u_1, u_2)^t$, let the divergence and curl of $\mathbf{u}$ be defined in the standard way: $\nabla \cdot \mathbf{u} = \partial_x u_1 + \partial_y u_2$ and $\nabla \times \mathbf{u} = \partial_x u_2 - \partial_y u_1$. Further, define the formal adjoint of the curl operator by

$$\nabla^\perp q = \begin{pmatrix} \partial_y q \\ -\partial_x q \end{pmatrix}.$$

We use standard notation for Sobolev spaces $H^k(\Omega)^d$, corresponding inner product $(\cdot,\cdot)_{k,\Omega}$, and norm $\|\cdot\|_{k,\Omega}$, for $k \geq 0$. We drop subscript Ω and superscript d when the domain and dimension are clear by context. Since $H^0(\Omega)$ coincides with $L^2(\Omega)$ we often denote $\|\cdot\|_0$ by $\|\cdot\|$. Define the subspaces of $L^2(\Omega)$ induced by the divergence and curl of $\mathbf{u}$ by

$$H(div) = \{\mathbf{u} \in L^2(\Omega) : \|\nabla \cdot \mathbf{u}\| < \infty\},$$
$$H(curl) = \{\mathbf{u} \in L^2(\Omega) : \|\nabla \times \mathbf{u}\| < \infty\}.$$

We also make use of the following general inequalities for nonnegative a and b:

$$|a|^2 + |b|^2 \leq |a+b|^2 \leq 2(|a|^2 + |b|^2). \tag{2.1}$$

Consider the function $f(r,\theta) = r^a$ in two dimensional polar coordinates. Assume that the origin lies on the boundary of domain

$$\Omega_w = \{(r,\theta) : 0 < r < R,\ 0 < \theta < \omega < 2\pi\},$$

as pictured in figure 2.1. By a direct computation it is clear that $f \in H^k(\Omega)$ only for $k < a+1$.

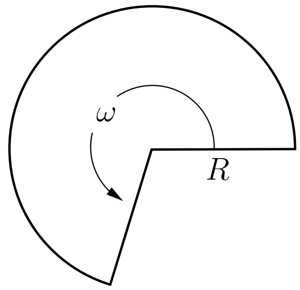


FIG. 2.1. *Simple wedge-shaped domain,* Ω_w.

Now, consider Poisson's equation on a domain in $\mathbb{R}^2$ with a corner of interior angle ω. It is well known that, for the case of Dirichlet or Neumann boundary conditions, the solutions of this boundary value problem may include those with radial part of the form $p \sim r^{\frac{\pi}{\omega}}$ in a local polar coordinate system centered at the corner. Thus, for the case of reentrant corners, $\omega > \pi$, the solution fails to be in $H^2(\Omega)$ and we say that the problem has a singularity (or singular solution). For problems with Dirichlet and Neumann boundary conditions meeting at the corner, solutions may have components of the form $p \sim r^{\frac{\pi}{(2\omega)}}$. Thus, for mixed boundary conditions, singularities may occur at corners with $\omega > \pi/2$. We now explore this issue in more detail.

Define the power of the singularity to be $\alpha = \pi/\omega$ for Dirichlet or Neumann boundary conditions and $\alpha = \pi/(2\omega)$ for mixed boundary conditions. The solution to Poisson's equation may be written as

$$p(r,\theta) = p_0(r,\theta) + s(r,\theta),$$

where $p_0(r,\theta) \in H^2(\Omega)$ and $s(r,\theta) \in H^{1+m}(\Omega)$ for $m < \alpha$. The singular part of the solution has the form

$$s(r,\theta) = r^{\alpha}(\kappa_1 \sin(\alpha\theta) + \kappa_2 \cos(\alpha\theta)),$$

where the values of κ_1 and κ_2 depend on boundary conditions (see [4, 5]).

For the FOSLS formulation of this problem, we may similarly decompose unknown $\mathbf{u} = \nabla p$ as

$$\mathbf{u}(r,\theta) = \mathbf{u}_0(r,\theta) + \nabla s(r,\theta),$$

where $\nabla s(r,\theta)$ has the form

$$\nabla s(r,\theta) = \alpha r^{\alpha-1} \begin{pmatrix} \kappa_1 \sin(\alpha-1)\theta + \kappa_2 \cos(\alpha-1)\theta \\ \kappa_1 \cos(\alpha-1)\theta - \kappa_2 \sin(\alpha-1)\theta \end{pmatrix}.$$

Thus, the unknown $\mathbf{u}(r,\theta)$ is in $H^k(\Omega)$ only for $k < \alpha$.

For example, consider Poisson's equation posed on the simple domain in figure 2.1. Let the solution to this boundary value problem in polar coordinates be $p = \chi(r) r^{\frac{2}{3}} \sin(2\theta/3)$, where $\chi(r)$ is a smooth transition function that is 1 on a platform near the origin and vanishes at the boundaries not adjacent to the origin. Then, $p = 0$ on $\partial\Omega$ and

$$\begin{aligned}\Delta p &= \frac{1}{r}\partial_r(r\partial_r p) + \frac{1}{r^2}\partial^2_{\theta\theta} p \\ &= \left(r^{\frac{2}{3}} \chi''(r) + \tfrac{7}{3} r^{\frac{-1}{3}} \chi'(r) \right) \sin(2\theta/3),\end{aligned}$$

and, thus, it is clear that $\Delta p \in L^2(\Omega)$, but $p \notin H^2(\Omega)$. We say this problem fails to provide *full lifting* of the data (from $L^2(\Omega)$ to $H^2(\Omega)$ e.g.). The solution, $\mathbf{u} = \nabla p$, is, thus, not in $H^1(\Omega)$.

3. Weighted-norm least squares. As before, let Ω be a domain with a corner of interior angle ω at the origin, and we may, without loss of generality, further assume $diam(\Omega) \leq 1$. For $f \in L^2(\Omega)$, let p satisfy

$$\begin{cases} -\Delta p = f, & \text{in} \quad \Omega, \\ p = 0, & \text{on} \quad \Gamma_D, \\ \mathbf{n} \cdot \nabla p = 0, & \text{on} \quad \Gamma_N, \end{cases} \tag{3.1}$$

where $\mathbf{n}$ is the outward unit normal to Ω and $\partial\Omega = \bar{\Gamma}_D \cup \bar{\Gamma}_N$. When this problem is H^2 regular, the normal FOSLS methodology is to introduce the new unknown, $\mathbf{u} = \nabla p$, and rewrite system (3.1) as

$$\begin{cases} \mathbf{u} - \nabla p = 0, & \text{in} \quad \Omega, \\ -\nabla \cdot \mathbf{u} = f, & \text{in} \quad \Omega, \\ \nabla \times \mathbf{u} = 0, & \text{in} \quad \Omega, \\ \boldsymbol{\tau} \cdot \mathbf{u} = 0, & \text{on} \quad \Gamma_D, \\ \mathbf{n} \cdot \mathbf{u} = 0, & \text{on} \quad \Gamma_N, \\ p = 0, & \text{on} \quad \Gamma_D, \\ \mathbf{n} \cdot \nabla p = 0, & \text{on} \quad \Gamma_N. \end{cases} \tag{3.2}$$

Here, $\boldsymbol{\tau}$ is the counter-clockwise unit tangental vector to Ω. Since this system can be posed completely in terms of $\mathbf{u}$, we may decouple the equations in (3.2), solve for $\mathbf{u}$ first, and then recover p from $\mathbf{u}$. To this end, define the two L^2-norm functionals,

$$\begin{aligned} G(\mathbf{u}; f) &= \|\nabla \cdot \mathbf{u} + f\|^2 + \|\nabla \times \mathbf{u}\|^2, \\ G_2(p; \mathbf{u}) &= \|\mathbf{u} - \nabla p\|^2, \end{aligned}$$

and the spaces,

$$\begin{aligned} \mathcal{V} &= \{\mathbf{v} \in H^1(\Omega) : \ \boldsymbol{\tau} \cdot \mathbf{v} = 0 \ \text{ on } \ \Gamma_D, \ \ \mathbf{n} \cdot \mathbf{v} = 0 \ \text{ on } \ \Gamma_N\}, \\ \mathcal{W} &= \{q \in H^1(\Omega) : \ q = 0 \ \text{ on } \ \Gamma_D\}. \end{aligned}$$

Thus, the two-stage solution process is to minimize $G(\mathbf{u}; f)$ over $\mathcal{V}$ and then, given the approximation to $\mathbf{u}$, minimize G_2 over $\mathcal{W}$:

$$\begin{aligned} &(1) \quad G(\mathbf{u}; f) = \inf_{\mathbf{v} \in \mathcal{V}} G(\mathbf{v}; f), \\ &(2) \quad G_2(p; \mathbf{u}) = \inf_{q \in \mathcal{W}} G_2(q; \mathbf{u}). \end{aligned} \tag{3.3}$$

The goal of the FOSLS methodology is, generally, to formulate functionals whose quadratic part is equivalent to the H^1 norm whenever possible. The second stage functional is H^1 equivalent and the solution we seek is always in H^1. The first stage functional, however, is not always H^1 equivalent. For domains with reentrant corners, there is no H^1 sequence of functions that converges to the solution in the $H(div) \cap H(curl)$ norm. To illustrate, consider the example above where $p = \chi(r) r^{\frac{2}{3}} \sin(2\theta/3)$ and $\mathbf{u} = \nabla p$. A simple computation reveals that $\nabla \cdot \mathbf{u}, \nabla \times \mathbf{u} \in L^2(\Omega)$, but $\mathbf{u} \notin H^1(\Omega)$.

Define the weighted functional by

$$G_w(\mathbf{u}; f) = \|w(\nabla \cdot \mathbf{u} + f)\|^2 + \|w \nabla \times \mathbf{u}\|^2, \tag{3.4}$$

where the weight function has the form $w = r^\beta$ for some $\beta > 0$.

Define the weighted Sobolev norm, $\|\cdot\|_{k,\beta}$, on Ω in terms of the standard L^2 norm, $\|\cdot\|_0$, by

$$\|q\|_{k,\beta} = (\sum_{|j| \le k} \|r^{\beta-k+j} D^j q\|_0^2)^{1/2}, \tag{3.5}$$

where D^j is the standard distributional derivative of order j. Similarly, define the weighted seminorm by

$$|q|_{k,\beta} = (\sum_{|j| = k} \|r^{\beta-k+j} D^j q\|_0^2)^{1/2} \tag{3.6}$$

and the associated weighted Sobolev space by

$$H^k_\beta(\Omega) = \{q : \|q\|_{k,\beta} < \infty\}. \tag{3.7}$$

Define the div-curl operator, L, and vector $\mathbf{f}$ by $L = \begin{pmatrix} \nabla \cdot \\ \nabla \times \end{pmatrix}$ and $\mathbf{f} = \begin{pmatrix} f \\ 0 \end{pmatrix}$. We may now write the weighted functional from (3.4) as

$$G_w(\mathbf{u}; f) = \|L\mathbf{u} - \mathbf{f}\|^2_{0,\beta}.$$

The weighted-norm least-squares minimization problem for the first-stage solution is then: find $\mathbf{u} \in \mathcal{V}$ such that

$$G_w(\mathbf{u}; f) = \inf_{\mathbf{v} \in \mathcal{V}} G_w(\mathbf{v}; f).$$

The second-stage solution for p remains as described above. We seek values of β that make $H^1(\Omega)$ dense in $H(div) \cap H(curl)$ in the weighted functional norm and result in the most accurate discretizations possible.

For the discrete problem, we may choose any finite dimensional subset of H^1 over which to minimize the weighted functional. Let $\mathcal{P}^h$ denote the space of C^0 piecewise polynomial (or tensor product) elements on triangles (or quadrilaterals) of meshsize h and $\mathcal{V}^h$ the subspace of $\mathcal{P}^h$ that satisfies the appropriate boundary conditions on Ω:

$$\mathcal{V}^h = \{\mathbf{v}^h \in \mathcal{P}^h : \ \boldsymbol{\tau} \cdot \mathbf{v}^h = 0 \ \text{ on } \ \Gamma_D, \ \ \mathbf{n} \cdot \mathbf{v}^h = 0 \ \text{ on } \ \Gamma_N\}.$$

The discrete weighted-norm least-squares minimization problem is, then, to minimize the discrete functional: find $\mathbf{u}^h \in \mathcal{V}^h$ such that

$$G_w(\mathbf{u}^h; f) = \min_{\mathbf{v}^h \in \mathcal{V}^h} G_w(\mathbf{v}^h; f). \tag{3.8}$$

By unweighting the equations near the singularity, the functional is freed from trying to approximate the solution (which is not in $H^1(\Omega)$) in the H^1 sense near the singularity. But, away from the singularity, the weighted functional retains the same character as the normal non-weighted functional. We now consider the choice of weight parameter β and its relation to weighted and nonweighted *a priori* error bounds on the approximated solution.

4. Theory and error bounds. In this section, we establish several theoretical results in weighted Sobolev spaces and error bounds for the weighted-norm method.

Here, we establish several Poincaré bounds in the domain Ω_w. We first prove a result for the scalar pure Neumann and pure Dirichlet problems, and then for the scalar mixed boundary condition problem. These results lead to a Poincaré inequality for the vector case.

Lemma 4.1 *Take $\Omega = \Omega_w$ and let $\beta > 0$, $\epsilon > 0$ and $\gamma = \beta - 3/2 - \epsilon$. Further, assume that $\gamma \neq -2$. If q is a scalar function in Ω that can be chosen to satisfy*

$$\iint_\Omega r^\gamma \, q(r,\theta)\, r\, dr\, d\theta = 0, \tag{4.1}$$

then

$$\|q\|_{0,\beta-1} \leq C(\epsilon) \|\nabla q\|_{0,\beta-\epsilon}, \tag{4.2}$$

where $C(\epsilon)$ depends on $\epsilon, \beta,$ and Ω and $C(\epsilon) \to \infty$ as $\epsilon \to 0$.

Proof. For any two points (r,θ) and (r_0,θ_0) in Ω, write $q(r,\theta)$ as

$$q(r,\theta) = q(r_0,\theta_0) + \int_{r_0}^{r} \frac{\partial q(\hat{r},\theta)}{\partial \hat{r}}\, d\hat{r} + \int_{\theta_0}^{\theta} \frac{\partial q(r_0,\hat{\theta})}{\partial \hat{\theta}}\, d\hat{\theta}.$$

Multiply both sides of the equation by $r_0^{\gamma+1}$ and integrate with respect to r_0 and θ_0 over Ω,

$$\begin{aligned}
&\frac{R^{\gamma+2}\omega}{\gamma+2} q(r,\theta) \\
&= \int_0^\omega \int_0^R \int_{r_0}^{r} r_0^{\gamma+1} \frac{\partial q(\hat{r},\theta)}{\partial \hat{r}}\, d\hat{r}\, dr_0\, d\theta_0 + \int_0^\omega \int_0^R \int_{\theta_0}^{\theta} r_0^{\gamma+1} \frac{\partial q(r_0,\hat{\theta})}{\partial \hat{\theta}}\, d\hat{\theta}\, dr_0\, d\theta_0 \\
&= \omega \int_0^r \int_0^{\hat{r}} r_0^{\gamma+1} \frac{\partial q(\hat{r},\theta)}{\partial \hat{r}}\, dr_0\, d\hat{r} - \omega \int_r^R \int_{\hat{r}}^R r_0^{\gamma+1} \frac{\partial q(\hat{r},\theta)}{\partial \hat{r}}\, dr_0\, d\hat{r} \\
&\quad + \int_0^R \int_0^\theta \int_0^{\hat{\theta}} r_0^{\gamma+1} \frac{\partial q(r_0,\hat{\theta})}{\partial \hat{\theta}}\, d\theta_0\, d\hat{\theta}\, dr_0 - \int_0^R \int_\theta^\omega \int_{\hat{\theta}}^\omega r_0^{\gamma+1} \frac{\partial q(r_0,\hat{\theta})}{\partial \hat{\theta}}\, d\theta_0\, d\hat{\theta}\, dr_0 \\
&= \frac{\omega}{\gamma+2} \int_0^R \hat{r}^{\gamma+2} \frac{\partial q(\hat{r},\theta)}{\partial \hat{r}}\, d\hat{r} - \frac{\omega R^{\gamma+2}}{\gamma+2} \int_r^R \frac{\partial q(\hat{r},\theta)}{\partial \hat{r}}\, d\hat{r} \\
&\quad + \int_0^R \int_0^\omega \hat{\theta}\, r_0^{\gamma+1} \frac{\partial q(r_0,\hat{\theta})}{\partial \hat{\theta}}\, d\hat{\theta}\, dr_0 - \omega \int_0^R \int_0^\omega r_0^{\gamma+1} \frac{\partial q(r_0,\hat{\theta})}{\partial \hat{\theta}}\, d\hat{\theta}\, dr_0.
\end{aligned}$$

By the triangle inequality we have that

$$\begin{aligned}
&\left|\frac{R^{\gamma+2}\omega}{\gamma+2} q(r,\theta)\right| \\
&\quad \le \frac{\omega}{\gamma+2}\int_0^R \hat{r}^{\gamma+2}\left|\frac{\partial q(\hat{r},\theta)}{\partial \hat{r}}\right| \, d\hat{r} + \frac{\omega R^{\gamma+2}}{\gamma+2}\int_r^R \left|\frac{\partial q(\hat{r},\theta)}{\partial \hat{r}}\right| \, d\hat{r} \\
&\qquad + 2\omega \int_0^R \int_0^\omega r_0^{\gamma+1}\left|\frac{\partial q(r_0,\hat{\theta})}{\partial \hat{\theta}}\right| \, d\hat{\theta}\, dr_0.
\end{aligned}$$

Now, squaring each side and using $(a+b+c)^2 \le 3(a^2+b^2+c^2)$ we get,

$$\begin{aligned}
|q(r,\theta)|^2 \le{}& \frac{3}{R^{2(\gamma+2)}}\left(\int_0^R \hat{r}^{\gamma+2}\left|\frac{\partial q(\hat{r},\theta)}{\partial \hat{r}}\right| \, d\hat{r}\right)^2 + 3\left(\int_r^R \left|\frac{\partial q(\hat{r},\theta)}{\partial \hat{r}}\right| \, d\hat{r}\right)^2 \\
&+ \frac{12(\gamma+2)^2}{R^{2(\gamma+2)}}\left(\int_0^R \int_0^\omega r_0^{\gamma+1}\left|\frac{\partial q(r_0,\hat{\theta})}{\partial \hat{\theta}}\right| \, d\hat{\theta}\, dr_0\right)^2.
\end{aligned} \tag{4.3}$$

Multiply each side of (4.3) by $r^{2\beta-1}$ and integrate with respect to r and θ over Ω. We consider each of the terms on the resulting right-hand side separately. First,

$$\begin{aligned}
&\int_0^\omega \int_0^R r^{2\beta-1}\left(\int_0^R \hat{r}^{\gamma+2}\left|\frac{\partial q(\hat{r},\theta)}{\partial \hat{r}}\right| \, d\hat{r}\right)^2 dr\, d\theta \\
&\quad \le R \int_0^\omega\int_0^R\int_0^R r^{2\beta-1}\,\hat{r}^{2\gamma+4}\left|\frac{\partial q(\hat{r},\theta)}{\partial \hat{r}}\right|^2 d\hat{r}\, dr\, d\theta \\
&\quad = \frac{R^{2\beta+1}}{2\beta}\int_0^\omega \int_0^R \hat{r}^{2\gamma+3}\left|\frac{\partial q(\hat{r},\theta)}{\partial \hat{r}}\right|^2 \hat{r}\, d\hat{r}\, d\theta \\
&\quad \le \frac{R^{2\beta+1}}{2\beta}\|\nabla q\|^2_{0,\gamma+\frac{3}{2}}.
\end{aligned}$$

We now consider the second term in (4.3). Since by the Schwarz inequality,

$$\begin{aligned}
&\left(\int_r^R \left|\frac{\partial q(\hat{r},\theta)}{\partial \hat{r}}\right| \, d\hat{r}\right)^2 \\
&\quad = \left(\int_r^R \hat{r}^{(\epsilon-1)/2}\hat{r}^{(1-\epsilon)/2}\left|\frac{\partial q(\hat{r},\theta)}{\partial \hat{r}}\right| \, d\hat{r}\right)^2 \\
&\quad \le \int_r^R \hat{r}^{\epsilon-1}\, d\hat{r}\int_r^R \hat{r}^{1-\epsilon}\left|\frac{\partial q(\hat{r},\theta)}{\partial \hat{r}}\right|^2 d\hat{r} \\
&\quad = \frac{R^\epsilon}{\epsilon}\int_r^R \hat{r}^{1-\epsilon}\left|\frac{\partial q(\hat{r},\theta)}{\partial \hat{r}}\right|^2 d\hat{r},
\end{aligned}$$

we can bound the second term:

$$
\begin{aligned}
&\int_0^\omega \int_0^R r^{2\beta-1} \left(\int_r^R \left| \frac{\partial q(\hat{r},\theta)}{\partial \hat{r}} \right| \, d\hat{r} \right)^2 dr\, d\theta \\
&\quad \le \frac{R^\epsilon}{\epsilon} \int_0^\omega \int_0^R \int_r^R r^{2\beta-1}\, \hat{r}^{1-\epsilon} \left| \frac{\partial q(\hat{r},\theta)}{\partial \hat{r}} \right|^2 d\hat{r}\, dr\, d\theta \\
&\quad = \frac{R^\epsilon}{\epsilon} \int_0^\omega \int_0^R \int_0^{\hat{r}} r^{2\beta-1}\, \hat{r}^{1-\epsilon} \left| \frac{\partial q(\hat{r},\theta)}{\partial \hat{r}} \right|^2 dr\, d\hat{r}\, d\theta \\
&\quad = \frac{R^\epsilon}{2\epsilon\beta} \int_0^\omega \int_0^R \hat{r}^{2\beta-\epsilon} \left| \frac{\partial q(\hat{r},\theta)}{\partial \hat{r}} \right|^2 \hat{r}\, d\hat{r}\, d\theta \\
&\quad = \frac{R^\epsilon}{2\epsilon\beta} \|\nabla q\|^2_{0,\beta-\frac{\epsilon}{2}} \\
&\quad \le \frac{R^\epsilon}{2\epsilon\beta} \|\nabla q\|^2_{0,\beta-\epsilon}.
\end{aligned}
$$

The third term can be bounded similarly:

$$
\begin{aligned}
&\int_0^\omega \int_0^R r^{2\beta-1} \left(\int_0^R \int_0^\omega r_0^{\gamma+1} \left| \frac{\partial q(r_0,\hat{\theta})}{\partial \hat{\theta}} \right| \, d\hat{\theta}\, dr_0 \right)^2 dr\, d\theta \\
&\quad \le \left(\int_0^R \int_0^\omega r^{2\beta-1}\, dr\, d\theta \right) R\omega \int_0^\omega \int_0^R r_0^{2\gamma+2} \left| \frac{\partial q(r_0,\hat{\theta})}{\partial \hat{\theta}} \right|^2 dr_0\, d\hat{\theta} \\
&\quad = \frac{R^{2\beta+1}\omega^2}{2\beta} \int_0^\omega \int_0^R r_0^{2\gamma+3} \left| \frac{1}{r_0} \frac{\partial q(r_0,\hat{\theta})}{\partial \hat{\theta}} \right|^2 r_0\, dr_0\, d\hat{\theta} \\
&\quad = \frac{R^{2\beta+1}\omega^2}{2\beta} \|\nabla q\|^2_{0,\gamma+\frac{3}{2}}.
\end{aligned}
$$

Putting the three terms together and substituting $\gamma = \beta - 3/2 - \epsilon$ we may now write the bound

$$
\begin{aligned}
&\int_0^\omega \int_0^R r^{2\beta-2} \left| q(r,\theta) \right|^2 r\, dr\, d\theta \\
&\quad \le \left(\frac{3R^{-2\epsilon}}{2\beta} + \frac{3R^\epsilon}{2\epsilon\beta} + \frac{6(\beta + \frac{1}{2} - \epsilon)^2 R^{-2\epsilon}\omega^2}{\beta} \right) \|\nabla q\|^2_{0,\beta-\epsilon},
\end{aligned}
$$

and the lemma follows by taking the square root of both sides. ∎

In what follows, let χ be a smooth function of r where $\chi = 1$ for $r < \eta$ and $\chi = 0$ for $r > 2\eta$. We take η to be sufficiently small to ensure that $supp(\chi) \subset \Omega$.

Lemma 4.2 *Take $\Omega = \Omega_w$ and let q be a scalar function in $H^1_\beta(\Omega)$, where $\beta > 0$. The following bound holds for χ as defined above:*

$$
\|\chi q\|_{0,\beta-1} \le \frac{1}{\beta} \|\nabla(\chi q)\|_{0,\beta}. \tag{4.4}
$$

Proof. Hardy's inequality for $f(t)$ defined for $t > 0$ with $\lim_{t\to 0} f(t) = 0$ gives (see [9]):

$$
\int_0^\infty \frac{f^2}{t^2}\, dt \le 4 \int_0^\infty |f'|^2\, dt. \tag{4.5}
$$

The lemma follows after a change of variables, $t = r^{-2\beta}$, a substitution $f(r) = \chi q(r,\theta)$ for fixed θ, and an integration on both sides with respect to θ. ■

Lemma 4.3 *Take $\Omega = \Omega_w$ and let either $q \in H^1_\beta(\Omega)$ with $q = 0$ on $\partial\Omega$ or $q \in H^1_\beta(\Omega)/\mathbb{R}$ with $n \cdot \nabla q = 0$ on $\partial\Omega$. Then*

$$\|q\|_{0,\beta-1} \le C\|\nabla q\|_{0,\beta} \tag{4.6}$$

for $\beta > 0$ where C depends only on Ω, and β.

Proof. First, if $q = 0$ on $\partial\Omega$, write $q = q(r,\theta)$ as

$$q(r,\theta) = \int_0^\theta \frac{\partial q(r,\hat\theta)}{\partial\hat\theta}\, d\hat\theta.$$

Square both sides and multiply by $r^{2\beta-1}$:

$$\begin{aligned} r^{2\beta-1}\,|q(r,\theta)|^2 &= r^{2\beta-1}\left|\int_0^\theta \frac{\partial q(r,\hat\theta)}{\partial\hat\theta}\, d\hat\theta\right|^2 \\ &\le r^{2\beta+1}\omega \int_0^\omega \left|\frac{1}{r}\frac{\partial q(r,\hat\theta)}{\partial\hat\theta}\right|^2 \, d\hat\theta. \end{aligned}$$

Integrate both sides with respect to r and θ over Ω:

$$\begin{aligned} \int_0^\omega\int_0^R r^{2\beta-2}\,|q(r,\theta)|^2\; r\,dr\,d\theta &\le \omega\int_0^\omega\int_0^{R_0} r^{2\beta+1}\int_0^\omega \left|\frac{1}{r}\frac{\partial q(r,\hat\theta)}{\partial\hat\theta}\right|^2 \, d\hat\theta\,dr\,d\theta \\ &\le \omega^2\int_0^{R_0}\int_0^\omega r^{2\beta}\left|\frac{1}{r}\frac{\partial q(r,\hat\theta)}{\partial\hat\theta}\right|^2 \, r\,d\hat\theta\,dr. \end{aligned}$$

The lemma follows since the right side is bounded by $C\|\nabla q\|^2_{0,\beta}$.

Now, if $q \in H^1_\beta(\Omega)/\mathbb{R}$ then it may be chosen to satisfy

$$\iint_\Omega r^\gamma\, q\, r\,dr\,d\theta = 0 \tag{4.7}$$

for γ chosen as in lemma 4.1. By the triangle inequality and lemma 4.2 we get

$$\begin{aligned} \|q\|_{0,\beta-1} &\le \|\chi\, q\|_{0,\beta-1} + \|(1-\chi)q\|_{0,\beta-1} \\ &\le C\|\nabla(\chi q)\|_{0,\beta} + \left(\int_0^\omega\int_\eta^R \left(\frac{r}{\eta}\right)^2 r^{2\beta-2}(1-\chi)^2\, q^2\, r\,dr\,d\theta\right)^{\frac{1}{2}} \\ &\le C\left(\|\nabla(q)\|_{0,\beta} + \|q\|_{0,\beta}\right). \end{aligned}$$

Apply lemma 4.1 with $\epsilon = 1$ to the $\|q\|_{0,\beta}$ term on the right side and the lemma follows. ■

For the problem with mixed boundary conditions, consider Ω_w partioned into subdomains $\Omega_0 = \{(r,\theta) : r \le \frac{1}{2}R_0, 0 \le \theta \le \omega\}$ and $\Omega_1 = \Omega\backslash\Omega_0$, as shown in figure 4.1.

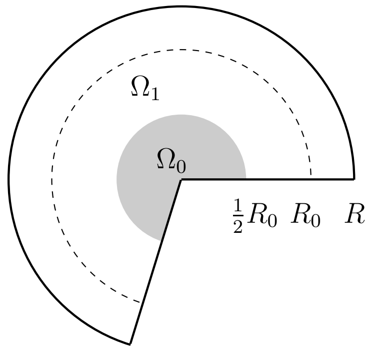


FIG. 4.1. *Wedge-shaped domain, Ω_w, partioned into subdomains Ω_0 and Ω_1.*

Lemma 4.4 *Consider domain $\Omega = \Omega_w$, as pictured in figure 4.1, and let $q \in H^1_\beta(\Omega)$ vanish on the line segment of $\partial\Omega$ corresponding to $\theta = 0$ and $r < R_0$. Then, there is a constant, C, dependent only on Ω, β, and R_0, such that*

$$\|q\|_{0,\beta-1} \le C\|\nabla q\|_{0,\beta} \tag{4.8}$$

for $\beta > 0$.

Proof. For points (r, θ) in Ω_0 we may derive the bound,

$$\|q\|_{0,\beta-1,\Omega_0} \le C\|\nabla q\|_{0,\beta,\Omega_0}, \tag{4.9}$$

completely analogous to the proof of lemma 4.3. Now, consider points (r, θ) in Ω_1. We may write $q = q(r, \theta)$ as

$$q(r,\theta) = \int_{\tilde{r}}^{r} \frac{\partial q(\hat{r},\theta)}{\partial \hat{r}}\, d\hat{r} + \int_0^{\theta} \frac{\partial q(\tilde{r},\hat{\theta})}{\partial \hat{\theta}}\, d\hat{\theta},$$

where the point $(\tilde{r}, 0)$ is on the part of $\partial\Omega_1$ where q vanishes. By the Schwarz inequality, the triangle inequality and inequality (2.1) we have the bound

$$|q(r,\theta)|^2 \le 2(R - \frac{1}{2}R_0) \int_{\tilde{r}}^{r} \left|\frac{\partial q(\hat{r},\theta)}{\partial \hat{r}}\right|^2 d\hat{r} + 2\omega \int_0^{\theta} \left|\frac{\partial q(\tilde{r},\hat{\theta})}{\partial \hat{\theta}}\right|^2 d\hat{\theta}. \tag{4.10}$$

We now expand the limits in the integrals, multiply each side by $r^{2\beta-1}$, integrate with respect to r over $(\frac{1}{2}R_0, R)$, integrate with respect to θ over $(0, \omega)$, and integrate with respect to $\tilde{r}$ over $(\frac{1}{2}R_0, R_0)$:

$$\begin{aligned}
&(\frac{1}{2}R_0) \int_0^{\omega} \int_{\frac{1}{2}R_0}^{R} r^{2\beta-1} |q(r,\theta)|^2 \; dr\, d\theta \\
&\quad \le 2(R - \frac{1}{2}R_0) \int_{\frac{1}{2}R_0}^{R_0} \int_0^{\omega} \int_{\frac{1}{2}R_0}^{R} \int_{\frac{1}{2}R_0}^{R} r^{2\beta-1} \left|\frac{\partial q(\hat{r},\theta)}{\partial \hat{r}}\right|^2 d\hat{r}\, dr\, d\theta\, d\tilde{r} \\
&\quad\quad + 2\omega \int_{\frac{1}{2}R_0}^{R_0} \int_0^{\omega} \int_{\frac{1}{2}R_0}^{R} \int_0^{\theta} \left|\frac{\partial q(\tilde{r},\hat{\theta})}{\partial \hat{\theta}}\right|^2 d\hat{\theta}\, dr\, d\theta\, d\tilde{r}.
\end{aligned} \tag{4.11}$$

We use the inequalities,

$$\frac{1}{2}R_0 \leq \tilde{r} \leq R_0, \quad \tilde{r} \leq \hat{r} \leq r \leq R,$$

to derive the following simple bounds:

$$1 \leq \left(\frac{2\hat{r}}{R_0}\right), \quad r \leq \left(\frac{2R}{R_0}\right)\hat{r}, \quad r \leq \left(\frac{2R}{R_0}\right)\tilde{r}. \tag{4.12}$$

By applying the bounds in (4.12) we may now write (4.11) as

$$\begin{aligned}
&(\frac{1}{2}R_0)\int_0^\omega \int_{\frac{1}{2}R_0}^R r^{2\beta-2}|q(r,\theta)|^2 r\ dr\, d\theta \\
&\leq (R-\frac{1}{2}R_0)^2 R_0 \int_0^\omega \int_{\frac{1}{2}R_0}^R \left(\frac{2\hat{r}}{R_0}\right)^2 \left(\frac{2R}{R_0}\right)^{2\beta-1} \hat{r}^{2\beta-1}\left|\frac{\partial q(\hat{r},\theta)}{\partial \hat{r}}\right|^2 d\hat{r}\, d\theta \\
&\quad + 2\omega^2(R-\frac{1}{2}R_0)\int_{\frac{1}{2}R_0}^{R_0}\int_0^\omega \left(\frac{2R}{R_0}\right)^{2\beta-1} \tilde{r}^{2\beta-1}\left|\frac{\partial q(\tilde{r},\hat{\theta})}{\partial\hat{\theta}}\right|^2 d\hat{\theta}\, d\tilde{r} \\
&\leq 2^{2\beta+1}(R-\frac{1}{2}R_0)^2 R_0^{-2\beta}R^{2\beta-1}\int_0^\omega\int_{\frac{1}{2}R_0}^R \hat{r}^{2\beta}\left|\frac{\partial q(\hat{r},\theta)}{\partial\hat{r}}\right|^2 \hat{r}\ d\hat{r}\, d\theta \\
&\quad + 2^{2\beta}\omega^2(R-\frac{1}{2}R_0)\left(\frac{R}{R_0}\right)^{2\beta-1}\int_{\frac{1}{2}R_0}^R\int_0^\omega \tilde{r}^{2\beta}\left|\frac{1}{\tilde{r}}\frac{\partial q(\tilde{r},\hat{\theta})}{\partial\hat{\theta}}\right|^2 \tilde{r}\ d\hat{\theta}\, d\tilde{r},
\end{aligned}$$

which directly implies

$$\|q\|_{0,\beta-1,\Omega_1} \leq C\|\nabla q\|_{0,\beta,\Omega_1}, \tag{4.13}$$

where

$$C = 2^{2\beta+1}(R-\frac{1}{2}R_0)R^{2\beta-1}R_0^{-2\beta-1}\left(2(R-\frac{1}{2}R_0)+\frac{\omega^2}{R_0}\right).$$

Combining inequalities (4.9) and (4.13) completes the lemma. ∎

We now consider a similar Poincaré inequality for the vector case. Again, consider $\Omega = \Omega_w$, where $\partial\Omega$ is partioned into Dirichlet and Neumann boundaries, Γ_D and Γ_N respectively. The following lemma is valid for the pure Dirichlet and Neumann cases and for the mixed boundary condition cases when Γ_D includes a part of the boundary adjacent to the origin and $\omega \neq \frac{3\pi}{2}$.

Lemma 4.5 *Take $\Omega = \Omega_w$ and let $\mathbf{u} \in H^1_\beta(\Omega)^2$ satisfy $\boldsymbol{\tau}\cdot\mathbf{u} = 0$ on Γ_D and $\mathbf{n}\cdot\mathbf{u} = 0$ on Γ_N. Assume that for the mixed boundary condition case that $\omega \neq 3\pi/2$. Then there is a constant, C, dependent only on Ω, β and the length of the segments of Γ_D and Γ_N adjacent to the origin, such that*

$$\|\mathbf{u}\|_{0,\beta-1} \leq C\|\nabla\mathbf{u}\|_{0,\beta} \tag{4.14}$$

for $\beta > 0$.

Proof. First, consider the case when $\boldsymbol{\tau} \cdot \mathbf{u} = 0$ on $\partial\Omega$. Denote the part of $\partial\Omega$ aligned with $\theta = 0$ as Γ_1 and the part of $\partial\Omega$ aligned with $\theta = \omega$ as Γ_2. Thus, $u_1 = 0$ on Γ_1 and $\tau_x u_1 + \tau_y u_2 = 0$ on Γ_2. Since u_1 and $\tau_x u_1 + \tau_y u_2$ satisfy the conditions in lemma 4.4, we may use

$$\|u_1\|_{0,\beta-1} \leq C\|\nabla u_1\|_{0,\beta}$$

and

$$\|\tau_x u_1 + \tau_y u_2\|_{0,\beta-1} \leq C\|\nabla(\tau_x u_1 + \tau_y u_2)\|_{0,\beta}.$$

Further, take $\tau_y \neq 0$, since $\tau_y = 0$ corresponds to either $\omega = \pi$, for which the result holds trivially since the boundary is smooth, or $\omega = 2\pi$, which we do not consider. Now,

$$\begin{aligned}
\|\mathbf{u}\|^2_{0,\beta-1} &= \|u_1\|^2_{0,\beta-1} + \|u_2\|^2_{0,\beta-1} \\
&= \|u_1\|^2_{0,\beta-1} + \frac{1}{\tau_y^2}\|\tau_x u_1 - \tau_x u_1 + \tau_y u_2\|^2_{0,\beta-1} \\
&\leq (1 + 2\frac{\tau_x^2}{\tau_y^2})\|u_1\|^2_{0,\beta-1} + \frac{2}{\tau_y^2}\|\tau_x u_1 + \tau_y u_2\|^2_{0,\beta-1} \\
&\leq C(\|\nabla u_1\|^2_{0,\beta} + \|\nabla(\tau_x u_1 + \tau_y u_2)\|^2_{0,\beta}) \\
&\leq C\|\nabla \mathbf{u}\|^2_{0,\beta}.
\end{aligned}$$

The case when $\mathbf{n} \cdot \mathbf{u} = 0$ on $\partial\Omega$ is analogous since $u_2 = 0$ on Γ_1 and $n_x u_1 + n_y u_2 = 0$ on Γ_2. Also, when $\omega \neq \frac{\pi}{2}, \frac{3\pi}{2}$, the case for mixed boundary conditions follows similarly using the result of lemma 4.4. The case for mixed boundary conditions when $\omega = \pi/2$, follows from appealing to symmetry in the the pure Dirichlet problem for $\omega = \pi$. ∎

Remark 4.6 *Lemma 4.5 can be directly extended to more generally shaped domains. The proof of the scalar Poincaré bounds in lemmas 4.1, 4.2, 4.3 and 4.4 are simplified when the domain has the shape of Ω_w with only one irregular boundary point. Since we are primarily interested in a local result, proving lemma 4.5 in the simple domain is sufficient for our purposes.*

Let $\mathcal{T}^h = \cup_{i=1}^N \tau_i$ be a quasi-uniform tesselation of polygonal domain Ω. Let $\mathcal{I}^h$ represent standard interpolation onto a piecewise polynomial finite element space of degree k. From finite element theory, we have the following interpolation bounds.

Lemma 4.7 *Let Ω be a polygonal domain. There exists a constant, C, independent of v, such that, for all $v \in H^m(\Omega)$,*

$$\|v - \mathcal{I}^h v\|_s \leq Ch^{m-s}|v|_m \tag{4.15}$$

for $0 \leq s \leq m$. When τ_i are triangles, $\mathcal{I}^h$ denotes interpolation by a piecewise polynomial of degree k, and $m \leq k+1$. When τ_i are quadrilaterals, $m \leq 2$ and $\mathcal{I}^h$ denotes bilinear interpolation.

Proof. See [1]. ∎

We now consider a weighted interpolation bound for functions on domains with a polygonal corner at the origin. Define the modified interpolation operator, $\mathcal{I}_0^h$, by

$$\mathcal{I}_0^h u|_\tau = \begin{cases} \mathcal{I}^h u = \sum_{i=0}^n u(a_i)\phi_i\ , & \text{if } \overline{\tau} \text{ does not intersect the origin,} \\ \sum_{i=1}^n u(a_i)\phi_i\ , & \text{if } \overline{\tau} \text{ intersects the origin,} \end{cases}$$

where $\mathcal{I}^h$ is a standard polynomial interpolation operator, ϕ_i are basis functions corresponding to the $n+1$ nodal points, a_i, and a_0 is the origin, $(0,0)$. Thus, the modified interpolation has a value of zero at the origin and resembles $\mathcal{I}^h$ away from the origin.

Lemma 4.8 *Let Ω be a polygonal domain. There exists a constant, C, independent of u, such that, for all $u \in H^m_\beta(\Omega)$ satisfying equation (4.8),*

$$\|u - \mathcal{I}^h_0 u\|_{1,\beta} \le C h^{m-1} \|u\|_{m,\beta}, \tag{4.16}$$

for $1 \le m \le k+1$ and $\beta > 0$, where $\mathcal{I}^h_0$ is the modified interpolation operator onto piecewise polynomials of degree k defined above.

Proof. We rewrite

$$\|u - \mathcal{I}^h_0 u\|^2_{1,\beta} \;=\; \sum_{\tau \in \mathcal{T}_h} \|u - \mathcal{I}^h_0 u\|^2_{1,\beta,\tau}$$

and consider $\|u - \mathcal{I}^h_0 u\|^2_{1,\beta,\tau}$ on each element τ. Define $K_0 = \{\tau \mid \bar{\tau} \cap (0,0) \ne \emptyset\}$ as the set of elements adjacent to the origin. On $\mathcal{T}^h \backslash K_0$, we have $h \le r_{min} \le r = \sqrt{x^2+y^2} \le r_{max} \le r_{min} + \sqrt{2}h$ with $r_{min} = \inf\{r | (x,y) \in \tau\}$ and $r_{max} = \sup\{r | (x,y) \in \tau\}$ in τ, and

$$\begin{aligned}
\|u - \mathcal{I}^h_0 u\|^2_{1,\beta,\tau} &= \|u - \mathcal{I}^h u\|^2_{1,\beta,\tau} \\
&= \int_\tau r^{2\beta} |\nabla(u - \mathcal{I}^h u)|^2 + r^{2(\beta-1)} |u - \mathcal{I}^h u|^2 d\tau \\
&\le r^{2\beta}_{max} \int_\tau |\nabla(u - \mathcal{I}^h u)|^2 d\tau + r^{2\beta}_{max} r^{-2}_{min} \int_\tau |u - \mathcal{I}^h u|^2 d\tau \\
&\le C r^{2\beta}_{max} h^{2(m-1)} |u|^2_{m,0,\tau} + C r^{2\beta}_{max} r^{-2}_{min} h^{2m} |u|^2_{m,0,\tau} \\
&= C r^{2\beta}_{max} h^{2(m-1)} (1 + r^{-2}_{min} h^2) |u|^2_{m,0,\tau} \le C r^{2\beta}_{max} h^{2(m-1)} |u|^2_{m,0,\tau} \\
&\le C h^{2(m-1)} r^{2\beta}_{max} r^{-2\beta}_{min} \int_\tau r^{2\beta} |D^m u|^2 d\tau \\
&\le C h^{2(m-1)} \left(\frac{r_{min} + \sqrt{2}h}{r_{min}} \right)^{2\beta} \int_\tau r^{2\beta} |D^m u|^2 d\tau \\
&\le C h^{2(m-1)} \int_\tau r^{2\beta} |D^m u|^2 d\tau.
\end{aligned}$$

We now consider the case for which $\tau \in K_0$. Let $\delta \in \mathcal{C}^\infty$ be a cut-off function defined by

$$\delta(r) = \begin{cases} 1, & \text{if} \quad r \le h/3, \\ 0, & \text{if} \quad r > 2h/3, \end{cases}$$

with $|\delta^{(m)}| \le c h^{-m}$, where $\delta^{(m)}$ is the m^{th} derivative of δ. By the triangle inequality,

$$\|u - \mathcal{I}^h_0 u\|_{1,\beta,\tau} \le \|\delta u - \mathcal{I}^h_0(\delta u)\|_{1,\beta,\tau} + \|(1-\delta)u - \mathcal{I}^h_0((1-\delta)u)\|_{1,\beta,\tau}. \tag{4.17}$$

By the definition of δ we have $\mathcal{I}_0^h((1-\delta)u) = \mathcal{I}^h((1-\delta)u)$ and $\mathcal{I}_0^h(\delta u) = 0$. For the second term in (4.17), we apply lemmas 4.4 and 4.7, and the properties in δ to obtain

$$
\begin{aligned}
&\|(1-\delta)u - \mathcal{I}_0^h((1-\delta)u)\|_{1,\beta,\tau}^2 = \|(1-\delta)u - \mathcal{I}^h((1-\delta)u)\|_{1,\beta,\tau}^2 \\
&\le C\|\nabla((1-\delta)u - \mathcal{I}^h((1-\delta)u))\|_{0,\beta,\tau}^2 \le Ch^{2\beta}\int_\tau |\nabla((1-\delta)u - \mathcal{I}^h((1-\delta)u))|^2 d\tau \\
&\le Ch^{2\beta}h^{2(m-1)}\int_\tau |D^m((1-\delta)u)|^2 d\tau \le Ch^{2(\beta+m-1)}\int_\tau \sum_{j=0}^m |D^{m-j}(1-\delta)D^j u|^2 d\tau \\
&\le Ch^{2(\beta+m-1)}\left(\int\int_{\frac{h}{3}}^{\frac{2h}{3}} \sum_{j=0}^{m-1} |h^{j-m}D^j u|^2 d\tau + \int\int_{\frac{h}{3}}^{r(\theta)} |(1-\delta)D^m u|^2 d\tau\right) \\
&\le Ch^{2(\beta+m-1)}\left(\sum_{j=0}^{m-1}\int\int_{\frac{h}{3}}^{\frac{2h}{3}} h^{-2\beta}r^{2(\beta+j-m)}|D^j u|^2 d\tau + \int\int_{\frac{h}{3}}^{r(\theta)} h^{-2\beta}r^{2\beta}|D^m u|^2 d\tau\right) \\
&= Ch^{2(m-1)}\sum_{j=0}^m \int_\tau r^{2(\beta+j-m)}|D^j u|^2 d\tau = Ch^{2(m-1)}\|u\|_{m,\beta,\tau}^2.
\end{aligned}
$$

Using the properties of δ results in a similar bound for the first term in (4.17):

$$
\begin{aligned}
&\|\delta u - \mathcal{I}_0^h(\delta u)\|_{1,\beta,\tau}^2 = \|\delta u\|_{1,\beta,\tau}^2 = \int_\tau r^{2\beta}|\nabla(\delta u)|^2 + r^{2(\beta-1)}|\delta u|^2 d\tau \\
&\le C\int_\tau r^{2\beta}(|\nabla\delta\cdot u|^2 + |\delta\nabla u|^2) + r^{2(\beta-1)}|\delta u|^2 d\tau \\
&\le C\int\int_{\frac{h}{3}}^{\frac{2h}{3}} r^{2\beta}h^{-2}|u|^2 d\tau + C\int\int_0^{\frac{2h}{3}} r^{2\beta}|\nabla u|^2 + r^{2(\beta-1)}|u|^2 d\tau \\
&\le C\int\int_{\frac{h}{3}}^{\frac{2h}{3}} r^{2(\beta-1)}|u|^2 d\tau + C\int\int_0^{\frac{2h}{3}} r^{2\beta}|\nabla u|^2 + r^{2(\beta-1)}|u|^2 d\tau \\
&\le C\int_\tau r^{2(m-1)}(r^{2(\beta-m+1)}|\nabla u|^2 + r^{2(\beta-m)}|u|^2)d\tau \le Ch^{2(m-1)}\|u\|_{m,\beta,\tau}^2.
\end{aligned}
$$

Thus we have

$$
\|u - \mathcal{I}_0^h u\|_{1,\beta}^2 = \sum_{\tau\in\mathcal{T}_h}\|u - \mathcal{I}_0^h u\|_{1,\beta,\tau}^2 \le Ch^{2(m-1)}\sum_{\tau\in\mathcal{T}_h}\|u\|_{m,\beta,\tau}^2 \le Ch^{2(m-1)}\|u\|_{m,\beta}^2,
$$

and the lemma follows. ■

Lemma 4.9 *Assume equation* (4.14) *holds in* Ω. *Then, for all* $\mathbf{u}^h \in \mathcal{V}^h$,

$$
\|\mathbf{u}^h\|_{0,\beta} \le Ch^{-\eta}\|\mathbf{u}^h\|_{0,\beta+\eta} \tag{4.18}
$$

for any real β *and any* $\eta > 0$.

Proof. Using lemma 4.5 and an inverse inequality, we may write

$$
\|\mathbf{u}^h\|_{0,\beta} \le C\|\nabla\mathbf{u}^h\|_{0,\beta+1} \le Ch^{-1}\|\mathbf{u}^h\|_{0,\beta+1},
$$

which establishes (4.18) for $\eta = 1$. Repeated application of this inequality thus validates (4.18) for any positive integer. Now consider

$$\begin{aligned}\|\mathbf{u}^h\|_{0,\beta}^2 &= \langle r^\beta \mathbf{u}^h, r^\beta \mathbf{u}^h\rangle = \langle r^{\beta-1/2}\mathbf{u}^h, r^{\beta+1/2}\mathbf{u}^h\rangle \\ &\le \|\mathbf{u}^h\|_{0,\beta-1/2}\|\mathbf{u}^h\|_{0,\beta+1/2} \le Ch^{-1}\|\mathbf{u}^h\|_{0,\beta+1/2}^2.\end{aligned}$$

Taking the square root establishes (4.18) for $\eta = 1/2$. Repeating these steps leads to (4.18) for all $\eta = \eta_n = k_n/2^{\ell_n}$ for any nonnegative integers k_n and ℓ_n. For any $\eta > 0$, choose a monotonically decreasing sequence, $\{\eta_n\}$, such that $\eta_n > \eta$ and $lim_{n\to\infty}|\eta_n - \eta| = 0$. Now, $g_n = (r^{\beta+\eta_n}\mathbf{u}^h)^2$ is a monotonically increasing function that converges to $g = (r^{\beta+\eta}\mathbf{u}^h)^2$ pointwise everywhere. Thus, by the Lebesgue monotone convergence theorem, we have

$$\|\mathbf{u}^h\|_{0,\beta+\eta}^2 = \int g dx = \lim_n \int g_n dx = \lim_n \|\mathbf{u}^h\|_{0,\beta+\eta_n}^2$$

and, therefore,

$$\begin{aligned}\|\mathbf{u}^h\|_{0,\beta} &= \lim_n \|\mathbf{u}^h\|_{0,\beta} \\ &\le \lim_n Ch^{-\eta_n}\|\mathbf{u}^h\|_{0,\beta+\eta_n} \\ &= (\lim_n Ch^{-\eta_n})(\lim_n \|\mathbf{u}^h\|_{0,\beta+\eta_n}) \\ &= Ch^{-\eta}\|\mathbf{u}^h\|_{0,\beta+\eta},\end{aligned}$$

which completes the proof. ∎

Consider the following scalar Poisson problem in Ω_w:

$$\begin{cases} \Delta p = f, & \text{in} \quad \Omega, \\ p = 0, & \text{on} \quad \Gamma_D, \\ \mathbf{n}\cdot\nabla p = 0, & \text{on} \quad \Gamma_N. \end{cases} \tag{4.19}$$

We refer to system (4.19) as the pure Dirichlet problem when $\partial\Omega = \Gamma_D$; the pure Neumann problem when $\partial\Omega = \Gamma_N$; and the mixed boundary condition problem when Γ_D includes the part of $\partial\Omega$ coinciding with one of either $\theta = 0$ or $\theta = \omega$, and $\Gamma_N = \partial\Omega\backslash\Gamma_D$ with $\Gamma_N \neq \emptyset$.

The following regularity results can be found in [16] and [15].

Lemma 4.10 *Assume $|1-\beta| < \pi/\omega$ for the pure Dirichlet problem, $0 < |1-\beta| < \pi/\omega$ for the pure Neumann problem and $|1-\beta| < \pi/2\omega$ for the mixed boundary condition problem. Then, for every $f \in H^0_\beta(\Omega)$, there exists a unique solution to (4.19), $p \in H^2_\beta(\Omega)$ for the pure Dirichlet and mixed boundary condition cases and $p \in H^2_\beta(\Omega)/\mathbb{R}$ for the pure Neumann problem. Moreover, there exists a constant, C, independent of p, such that*

$$\|p\|_{2,\beta} \le C\|f\|_{0,\beta}. \tag{4.20}$$

Proof. See Chapter 1 of [16] for the Dirichlet and Neumann problems and Chapter 2 of [15] for the mixed boundary problem. ∎

Define the subspace of functions in $H^1_\beta(\Omega)$ satisfying the appropriate boundary conditions by

$$\mathcal{V}_\beta = \{\mathbf{v} \in H^1_\beta(\Omega) : \ \boldsymbol{\tau}\cdot\mathbf{v} = 0 \ \text{ on } \ \Gamma_D, \ \ \mathbf{n}\cdot\mathbf{v} = 0 \ \text{ on } \ \Gamma_N\}.$$

We now prove a regularity result for functions in $\mathcal{V}_\beta$. Recall that the power of the singularity is defined as $\alpha = \pi/\omega$ for Dirichlet or Neumann boundary conditions and $\alpha = \pi/(2\omega)$ for mixed boundary conditions.

Lemma 4.11 *Consider domain $\Omega = \Omega_w$. Then there is a positive constant, C, independent of $\mathbf{u}$, such that, for $|1-\beta| < \alpha$, the following bound holds for all $\mathbf{u} \in \mathcal{V}_\beta$:*

$$\|\mathbf{u}\|_{1,\beta} \le C\|L\mathbf{u}\|_{0,\beta}. \tag{4.21}$$

Proof. From lemmma 4.10 we know that any $\mathbf{u} \in \mathcal{V}_\beta$ has the decomposition

$$\mathbf{u} = \nabla\phi + \nabla^\perp\psi, \tag{4.22}$$

where $\phi, \psi \in H^2_\beta(\Omega)$ satisfy

$$\begin{cases} \Delta\phi = \nabla\cdot\mathbf{u}, & \text{in} \quad \Omega, \\ \quad \phi = 0, & \text{on} \quad \partial\Omega, \end{cases} \tag{4.23}$$

and

$$\begin{cases} \quad \Delta\psi = \nabla\times\mathbf{u}, & \text{in} \quad \Omega, \\ \mathbf{n}\cdot\nabla\psi = 0, & \text{on} \quad \partial\Omega. \end{cases} \tag{4.24}$$

Then, by applying lemma 4.10 to problems (4.23) and (4.24) we have

$$\begin{aligned} \|\mathbf{u}\|_{1,\beta} &= \|\nabla\phi + \nabla^\perp\psi\|_{1,\beta} \le \|\nabla\phi\|_{1,\beta} + \|\nabla^\perp\psi\|_{1,\beta} \\ &\le \|\phi\|_{2,\beta} + \|\psi\|_{2,\beta} \le C(\|\nabla\cdot\mathbf{u}\|_{0,\beta} + \|\nabla\times\mathbf{u}\|_{0,\beta}) \le C\|L\mathbf{u}\|_{0,\beta}, \end{aligned}$$

which completes the proof. ∎

Define an *irregular boundary point* of polygonal domain Ω to be a point on $\partial\Omega$ where interior angle ω satisfies $\omega > \pi$ when Dirichlet or Neumann boundary conditions are applied on both sides of the point or $\omega > \pi/2$ when one Dirichlet boundary and a Neumann boundary meet at the corner. We now present error bounds for the numerical solution in weighted and unweighted norms.

Theorem 4.12 *Let Ω be a polygonal domain with one irregular boundary point of interior angle ω and let $\mathbf{f} \in L^2(\Omega)$. Suppose $\mathbf{u} \in \mathcal{V}$ satisfies $L\mathbf{u} = \mathbf{f}$. If $\mathbf{u}^h \in \mathcal{V}^h$ is chosen to minimize the weighted functional,*

$$G_w(\mathbf{u}^h; \mathbf{f}) = \|L\mathbf{u}^h - \mathbf{f}\|^2_{0,\beta} = \inf_{\mathbf{v}^h \in \mathcal{V}^h} \|L\mathbf{v}^h - \mathbf{f}\|^2_{0,\beta},$$

for $|1-\beta| < \alpha$, then the approximation error, $\mathbf{u} - \mathbf{u}^h$, satisfies the following bounds:

$$\|\mathbf{u} - \mathbf{u}^h\|_{1,\beta} \le Ch^{\alpha+\beta-1}\|\mathbf{u}\|_{\alpha+\beta,\beta}, \tag{4.25}$$

$$G_w(\mathbf{u} - \mathbf{u}^h; \mathbf{0})^{1/2} \le Ch^{\alpha+\beta-1}\|\mathbf{u}\|_{\alpha+\beta,\beta}, \tag{4.26}$$

$$\|\mathbf{u} - \mathbf{u}^h\|_0 \le Ch^{\alpha}(|\mathbf{u}|_\alpha + \|\mathbf{u}\|_{\alpha+\beta,\beta} + \|\mathbf{u}\|_{\alpha-\beta+2,2-\beta}), \tag{4.27}$$

for $\alpha + \beta \le k+1$, where k is the degree of the piecewise polynomial elements in $\mathcal{V}^h$.

Proof. By lemmas 4.11 and 4.8, we have

$$\begin{aligned}\|\mathbf{u}-\mathbf{u}^h\|_{1,\beta} &\le C\|L(\mathbf{u}-\mathbf{u}^h)\|_{0,\beta} \le C\|L(\mathbf{u}-\mathcal{I}_0^h\mathbf{u})\|_{0,\beta}\\ &\le C\|\mathbf{u}-\mathcal{I}_0^h\mathbf{u}\|_{1,\beta} \le Ch^{\alpha+\beta-1}\|\mathbf{u}\|_{\alpha+\beta,\beta},\end{aligned}$$

which establishes both (4.25) and (4.26) since we may write

$$\|L(\mathbf{u}-\mathbf{u}^h)\|_{0,\beta} = G_w(\mathbf{u}-\mathbf{u}^h;\mathbf{0})^{1/2}.$$

Note that lemmas 4.11 and lemma 4.8 are satisfied for $|1-\beta|<\alpha$ and $\alpha+\beta\le 2$. We also have

$$\|\mathbf{u}^h-\mathcal{I}_0^h\mathbf{u}\|_{1,\beta} \le \|\mathbf{u}-\mathbf{u}^h\|_{1,\beta} + \|\mathbf{u}-\mathcal{I}_0^h\mathbf{u}\|_{1,\beta} \le Ch^{\alpha+\beta-1}\|\mathbf{u}\|_{\alpha+\beta,\beta}. \tag{4.28}$$

We now consider the bound on $\|\mathbf{u}-\mathbf{u}^h\|_0$. Let $K_0=\{\tau \mid \bar{\tau}\cap(0,0)\neq\emptyset\}$ and $K_1=\mathcal{T}^h\backslash K_0$. We may write

$$\|\mathbf{u}-\mathbf{u}^h\|_0^2 = \sum_{\tau\in K_0}\|\mathbf{u}-\mathbf{u}^h\|_{0,0,\tau}^2 + \sum_{\tau\in K_1}\|\mathbf{u}-\mathbf{u}^h\|_{0,0,\tau}^2.$$

When $\tau\in K_1$, we have $\mathcal{I}_0^h=\mathcal{I}^h$ and, for $\beta\ge 1$, we apply the triangle inequality and lemmas 4.7, 4.9 and 4.5 to get

$$\begin{aligned}\|\mathbf{u}-\mathbf{u}^h\|_{0,0,\tau} &\le \|\mathbf{u}-\mathcal{I}^h\mathbf{u}\|_{0,0,\tau} + \|\mathbf{u}^h-\mathcal{I}^h\mathbf{u}\|_{0,0,\tau}\\ &\le C(h^\alpha|\mathbf{u}|_{\alpha,\tau} + h^{1-\beta}\|\mathbf{u}^h-\mathcal{I}^h\mathbf{u}\|_{0,\beta-1,\tau})\\ &\le C(h^\alpha|\mathbf{u}|_{\alpha,\tau} + h^{1-\beta}\|\mathbf{u}^h-\mathcal{I}^h\mathbf{u}\|_{1,\beta,\tau}).\end{aligned} \tag{4.29}$$

Similarly, if $\beta<1$ we have

$$\begin{aligned}\|\mathbf{u}-\mathbf{u}^h\|_{0,0,\tau} &\le \|\mathbf{u}-\mathcal{I}^h\mathbf{u}\|_{0,0,\tau} + \|\mathbf{u}^h-\mathcal{I}^h\mathbf{u}\|_{0,0,\tau}\\ &\le C(h^\alpha|\mathbf{u}|_{\alpha,0,\tau} + h^{\beta-1}\|\mathbf{u}^h-\mathcal{I}^h\mathbf{u}\|_{0,1-\beta,\tau})\\ &\le C(h^\alpha|\mathbf{u}|_{\alpha,0,\tau} + h^{\beta-1}\|\mathbf{u}^h-\mathcal{I}^h\mathbf{u}\|_{1,2-\beta,\tau}).\end{aligned} \tag{4.30}$$

When $\tau\in K_0$, we have $r\le Ch$ and for $\beta\ge 1$ we apply the triangle inequality and lemmas 4.9 and 4.5 to get

$$\begin{aligned}\|\mathbf{u}-\mathbf{u}^h\|_{0,0,\tau} &\le \|\mathbf{u}-\mathcal{I}_0^h\mathbf{u}\|_{0,0,\tau} + \|\mathbf{u}^h-\mathcal{I}_0^h\mathbf{u}\|_{0,0,\tau}\\ &\le C(h^{\beta-1}\|\mathbf{u}-\mathcal{I}_0^h\mathbf{u}\|_{0,1-\beta,\tau} + h^{1-\beta}\|\mathbf{u}^h-\mathcal{I}_0^h\mathbf{u}\|_{0,\beta-1,\tau})\\ &\le C(h^{\beta-1}\|\mathbf{u}-\mathcal{I}_0^h\mathbf{u}\|_{1,2-\beta,\tau} + h^{1-\beta}\|\mathbf{u}^h-\mathcal{I}_0^h\mathbf{u}\|_{1,\beta,\tau})\end{aligned} \tag{4.31}$$

Similarly, if $\beta<1$ we have

$$\begin{aligned}\|\mathbf{u}-\mathbf{u}^h\|_{0,0,\tau} &\le \|\mathbf{u}-\mathcal{I}_0^h\mathbf{u}\|_{0,0,\tau} + \|\mathbf{u}^h-\mathcal{I}_0^h\mathbf{u}\|_{0,0,\tau}\\ &\le C(h^{1-\beta}\|\mathbf{u}-\mathcal{I}_0^h\mathbf{u}\|_{0,\beta-1,\tau} + h^{\beta-1}\|\mathbf{u}^h-\mathcal{I}_0^h\mathbf{u}\|_{0,1-\beta,\tau})\\ &\le C(h^{1-\beta}\|\mathbf{u}-\mathcal{I}_0^h\mathbf{u}\|_{1,\beta,\tau} + h^{\beta-1}\|\mathbf{u}^h-\mathcal{I}_0^h\mathbf{u}\|_{1,2-\beta,\tau})\end{aligned} \tag{4.32}$$

Now, considering the cases above for $\beta\ge 1$, we combine inequalities (4.29), (4.31), (4.28) and (4.25) to get

$$\begin{aligned}\|\mathbf{u}-\mathbf{u}^h\|_0 &\le C(h^\alpha|\mathbf{u}|_\alpha + h^{\beta-1}\|\mathbf{u}-\mathcal{I}_0^h\mathbf{u}\|_{1,2-\beta} + h^{1-\beta}\|\mathbf{u}^h-\mathcal{I}_0^h\mathbf{u}\|_{1,\beta})\\ &\le C(h^\alpha|\mathbf{u}|_\alpha + h^{\beta-1}h^{\alpha-\beta+1}\|\mathbf{u}\|_{\alpha+2-\beta,2-\beta} + h^{1-\beta}h^{\alpha+\beta-1}\|\mathbf{u}\|_{\alpha+\beta,\beta})\\ &\le Ch^\alpha(|\mathbf{u}|_\alpha + \|\mathbf{u}\|_{\alpha+2-\beta,2-\beta} + \|\mathbf{u}\|_{\alpha+\beta,\beta}).\end{aligned} \tag{4.33}$$

The case when $\beta<1$ follows analogously which establishes the lemma. ∎

Remark 4.13 *For the optimal finite element convergence of $O(h)$ with respect to the weighted functional and H^1 norms, we select $\beta = 2-\alpha$. But theorem 4.12 also requires that $\beta < 1+\alpha$. Thus, when $\alpha \in [1/2, 1)$, we may use a weighting with $\beta = 2-\alpha$ and expect optimal rates, but when $\alpha \in (0, 1/2)$, our theory only guarantees at best $O(2\alpha)$ convergence using $\beta = 1+\alpha$. Numerical results, however, indicate that values of β larger than the theory allows can be used to recover optimal rates. We explore this in the next section.*

5. Computational results. In this section, we present some numerical examples of the weighted-norm procedure to validate the error bounds in the previous section.

As a test problem, we minimize the weighted functional on the following L-shaped domain: $\Omega = (-0.5, 0.5)^2 \setminus [0, 0.5) \times (-0.5, 0]$, which yields $\alpha = \pi/\omega = 2/3$. Function $\mathbf{f}$ is chosen so that the solution of this test problem is $\mathbf{u} = \nabla(\chi(r) r^{\frac{2}{3}} \sin(2\theta/3))$, where $\chi(r) = 1$ for $r < 1/8$, $\chi(r) = 0$ for $r > 3/8$, and $\chi(r)$ is C^2 smooth. Again, note that $\mathbf{f} \in L^2(\Omega)$ but $\mathbf{u} \notin H^1(\Omega)$.

Define the following measures of the accuracy of the computed solution, $\mathbf{u}^h$:

nonweighted functional norm	$G^{1/2} = (\lVert \nabla \cdot \mathbf{u}^h - f \rVert_0^2 + \lVert \nabla \times \mathbf{u}^h \rVert_0^2)^{1/2}$,
nonweighted L^2 norm of the error	$\epsilon^0 = \lVert \mathbf{u} - \mathbf{u}^h \rVert_0$,
nonweighted H^1 seminorm of the error	$\epsilon^1 = \lvert \mathbf{u} - \mathbf{u}^h \rvert_1$,
weighted functional norm	$G_w^{1/2} = G_w(\mathbf{u}^h; f)^{1/2}$,
weighted L^2 norm of the error	$\epsilon_w^0 = \lVert \mathbf{u} - \mathbf{u}^h \rVert_{0,\beta}$,
weighted H^1 seminorm of the error	$\epsilon_w^1 = \lvert \mathbf{u} - \mathbf{u}^h \rvert_{1,\beta}$.

Since $\alpha = 2/3$, we choose the optimal weight parameter, $\beta = 2 - \alpha = 4/3$, for our computations. Table 5.1 summarizes discretization error and convergence rates for $\beta = 4/3$.

Asymptotic convergence rates in Ω are found to be approximately $O(h)$ for $G_w^{1/2}$ and ϵ_w^1, $O(h^2)$ for ϵ_w^0 and $O(h^{\frac{2}{3}})$ for ϵ_0. The approximation does not converge in either the ϵ_1 or $G^{1/2}$ measures since $\mathbf{u} \notin H^1(\Omega)$.

To distinguish between behavior near to and away from the singularity, we consider the error of the solution above on a partitioning of Ω. Define $\Omega_0 = \Omega \cap (\frac{3}{8}, \frac{5}{8})^2$ and $\Omega_1 = \Omega \backslash \Omega_0$; see figure 5.1.

Table 5.2 summarizes the asymptotic discretization accuracy obtained at the finest mesh size in subdomains Ω_0 and Ω_1. Away from the singularity we observe optimal accuracy in all measures. As expected, near the singularity, the solution fails to converge in the nonweighted functional and H^1 norms. The nonweighted L^2 error achieves accuracy of approximately $O(h^{\frac{2}{3}})$ near the singularity.

Figure 5.2 shows the first component of the exact solution, u_1, and the standard FOSLS approximation u_1^h. Figure 5.3 shows the error of the first component of the approximated solution for the standard FOSLS and the weighted-norm FOSLS methods. We see that the error in the approximation in standard FOSLS is highest near the singularity, but remains large even away from the corner point. In the weighted-norm FOSLS implementation, the error remains large near the singularity, as we expect, but is now concentrated only near the corner point. The pollution effect is removed by the weighting procedure.

h	$G_w^{1/2}$	Ratio	Rate	ϵ_w^1	Ratio	Rate
8^{-1}	5.52			3.81		
16^{-1}	4.34	1.27	0.35	1.47	2.59	1.37
32^{-1}	2.34	1.85	0.89	6.66e-01	2.21	1.14
64^{-1}	1.19	1.97	0.98	2.97e-01	2.24	1.16
128^{-1}	5.98e-01	1.99	0.99	1.41e-01	2.11	1.08
256^{-1}	3.00e-01	1.99	0.99	6.74e-02	2.09	1.06
512^{-1}	1.50e-01	2.00	1.00	3.31e-02	2.04	1.03

h	ϵ_w^0	Ratio	Rate	ϵ^0	Ratio	Rate
8^{-1}	3.08e-01			3.72e-01		
16^{-1}	1.35e-01	2.28	1.19	1.93e-01	1.93	0.95
32^{-1}	4.07e-02	3.32	1.73	8.93e-02	2.16	1.11
64^{-1}	1.11e-02	3.67	1.88	4.93e-02	1.81	0.86
128^{-1}	2.98e-03	3.72	1.90	3.00e-02	1.64	0.71
256^{-1}	7.84e-04	3.80	1.93	1.87e-02	1.60	0.68
512^{-1}	2.06e-04	3.81	1.93	1.18e-02	1.58	0.66

TABLE 5.1
Convergence of discretization error for weighted-norm FOSLS.

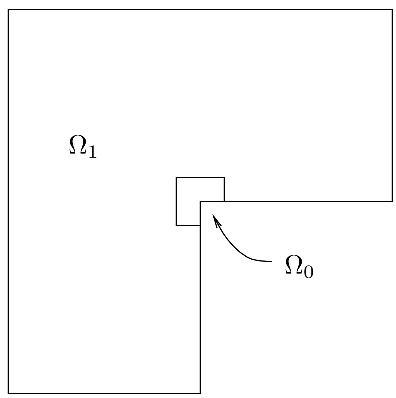
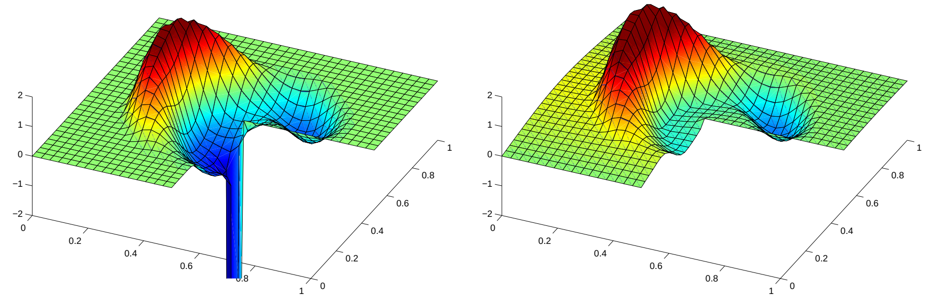


FIG. 5.1. *L-shaped domain Ω and subdomains Ω_0 and Ω_1.*

	$G_w^{1/2}$	$G^{1/2}$	ϵ_w^1	ϵ^1	ϵ_w^0	ϵ^0
Ω_1	$O(h)$	$O(h)$	$O(h)$	$O(h)$	$O(h^2)$	$O(h^2)$
Ω_0	$O(h)$	$O(1)$	$O(h)$	$O(1)$	$O(h^2)$	$O(h^{\frac{2}{3}})$
Ω	$O(h)$	$O(1)$	$O(h)$	$O(1)$	$O(h^2)$	$O(h^{\frac{2}{3}})$

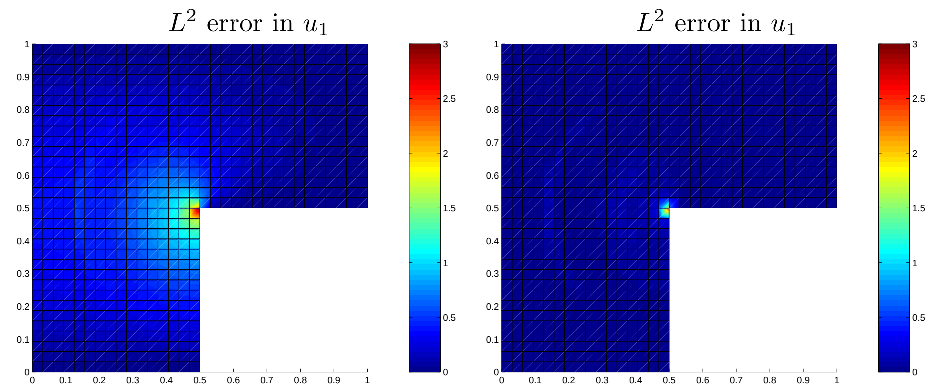
TABLE 5.2
Accuracy in Ω_0, Ω_1, and Ω with $\beta = 2 - \alpha$.

There are many boundary value problems not directly covered by the theory presented here that are of interest. For example, Poisson's equation with mixed boundary conditions on the domain used above has a value of $\alpha = 1/3$. To recover optimal convergence for this problem, the weighted-norm method requires a value of β



(a) Exact component u_1. (b) Component u_1^h computed by standard FOSLS.

FIG. 5.2. *Exact solution component u_1 and solution component u_1^h approximated by standard FOSLS on the $h = 32^{-1}$ mesh.*



(a) Standard FOSLS. (b) Weighted-norm FOSLS, $\beta = 2 - \alpha$.

FIG. 5.3. *Reduction of the pollution effect by the weighted-norm procedure. Each plot is the error of solution component u_1^h on the $h = 32^{-1}$ mesh.*

larger than theorem 4.12 allows. In other elliptic equations (e.g., Stokes or the linear elasticity equations), the value of α is generally smaller than for Poisson's equation for the same domain and boundary condition type. In each of these cases, a larger β value is necessary for optimal convergence. This leads us to consider using larger β than the theory allows.

Consider the same example problem as above on uniform mesh sizes of $h = 1/8, 1/16, ..., 1/512$, and values of β ranging from 1/3 to 23/6.

Figure 5.4 plots the convergence rate at the finest level for: the weighted functional norm, $G_w^{1/2}$; the weighted L^2 norm, ϵ_w^0; and the L^2 norm, ϵ^0. While the functional norm retains optimal accuracy for large values of β, the solution fails to converge in the weighted and nonweighted L^2 measures for $\beta \gtrsim 3$. This indicates that, although the weighted-norm approach seems to be more robust than the theory allows, large

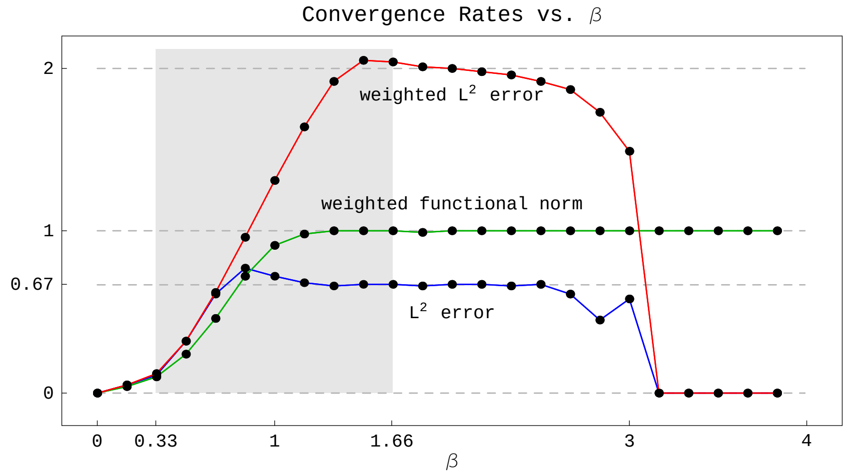


FIG. 5.4. *Convergence rates versus β. The shaded region indicates values of β for which the assumptions of theorem 4.12 are satisfied.*

values of β should still be used with caution.

The method presented here is applicable to a wide range of problems and provides an efficient alternative to more specialized techinques for treating singularities in boundary value problems. Further numerical results for other problems including systems and in three dimensions can be seen in a companion paper.

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