
5.1 Energy norm. Assume A is symmetric positive definite. As defined in this chapter, the A-energy inner product and the A-energy norm are given by

$$(\mathbf{u}, \mathbf{v})_A = (A\mathbf{u}, \mathbf{v}) \quad \text{and} \quad \|\mathbf{u}\|_A^2 = (\mathbf{u}, \mathbf{u})_A. \tag{1}$$

(a) Show that these are acceptable definitions for an inner product and a norm.

We will first show that $(.,.)_A$ satisfies the requirements for an inner product:

i. Symmetry. $(\mathbf{u}, \mathbf{v})_A = (A\mathbf{u}, \mathbf{v}) = (\mathbf{u}, A^T\mathbf{v}) = (\mathbf{u}, A\mathbf{v}) = (A\mathbf{v}, \mathbf{u}) = (\mathbf{v}, \mathbf{u})_A$, by the symmetry of A and the symmetry of the l_2 inner product.

ii. Multiplicative Linear. $(\alpha\mathbf{u}, \mathbf{v})_A = (A\alpha\mathbf{u}, \mathbf{v}) = (\alpha A\mathbf{u}, \mathbf{v}) = \alpha(A\mathbf{u}, \mathbf{v})$, by the multiplicative linearity of A.

iii. Additive Linear. $(\mathbf{u}, \mathbf{v}+\mathbf{w})_A = (A\mathbf{u}, \mathbf{v}+\mathbf{w}) = (A\mathbf{u}, \mathbf{v})+(A\mathbf{u}, \mathbf{w}) = (\mathbf{u}, \mathbf{v})_A + (\mathbf{u}, \mathbf{w})_A$, by the additive linearity of the l_2 inner product.

iv. Positive. $(\mathbf{u}, \mathbf{u})_A = (A\mathbf{u}, \mathbf{u}) \geq 0$, with equality only occurring for $\mathbf{u} = \mathbf{0}$, due to A being symmetric positive definite.

This inner product $(.,.)_A$, as does any inner product, induces a norm $\|\mathbf{u}\|_A = \sqrt{(\mathbf{u}, \mathbf{u})_A}$ that satisfies all norm requirements.

(b) Show that $\|\mathbf{r}\|_2 = \|\mathbf{e}\|_{A^2}$.

Using the definition of the norm squared, the residual equation $A\mathbf{e} = \mathbf{r}$, and the symmetry of A, we have

$$\|\mathbf{r}\|_2^2 = (\mathbf{r}, \mathbf{r}) = (A\mathbf{e}, A\mathbf{e}) = (A^TA\mathbf{e}, \mathbf{e}) = (A^2\mathbf{e}, \mathbf{e}) = \|\mathbf{e}\|_{A^2}^2. \tag{2}$$

The desired result holds because both $\|\mathbf{r}\|_2$ and $\|\mathbf{e}\|_{A^2}$ are positive.

(c) The error norm $\|\mathbf{e}\|_2$ is generally not computable. Is $\|\mathbf{e}\|_A$ computable? Is $\|\mathbf{e}\|_{A^2}$ computable?

The value is $\|\mathbf{e}\|_2$ is not *reasonably* computable because the error would have to be available to compute this norm. If the error was available, we would be done solving the problem and not discussing what the error is anymore! The residual $\mathbf{r} = \mathbf{f} - A\mathbf{v}$ is always available. For the A-energy norm of the error $\|\mathbf{e}\|_A$, we have

$$\|\mathbf{e}\|_A = \sqrt{(A\mathbf{e}, \mathbf{e})} = \sqrt{(\mathbf{r}, \mathbf{e})}, \tag{3}$$

which again is not reasonably computable because the error is not available. The A^2 norm, however, is computable because it can be written entirely dependent on the residual, $\|\mathbf{e}\|_{A^2} = \|\mathbf{r}\|_2$.

5.2 FMG error analysis. A key step in the FMG error analysis is showing that

$$\|I_{2h}^h \mathbf{u}^{2h} - I_{2h}^h \mathbf{v}^{2h}\|_{A^h} = c\|\mathbf{u}^{2h} - \mathbf{v}^{2h}\|_{A^{2h}} \tag{4}$$

(For some constant c). Use the Galerkin property and the property of inner products that

$$(B\mathbf{u}, \mathbf{v}) = (\mathbf{u}, B^T\mathbf{v}) \tag{5}$$

to prove this equality for any two coarse-grid vectors.

After some algebra, we have the result,

$$\begin{aligned}
\|I_{2h}^h \mathbf{u}^{2h} - I_{2h}^h \mathbf{v}^{2h}\|_{A^h}^2 &= (A^h(I_{2h}^h \mathbf{u}^{2h} - I_{2h}^h \mathbf{v}^{2h}), I_{2h}^h \mathbf{u}^{2h} - I_{2h}^h \mathbf{v}^{2h}) \\
&= (A^h I_{2h}^h(\mathbf{u}^{2h} - \mathbf{v}^{2h}), I_{2h}^h(\mathbf{u}^{2h} - \mathbf{v}^{2h})) \\
&= ((I_{2h}^h)^T A^h I_{2h}^h(\mathbf{u}^{2h} - \mathbf{v}^{2h}), \mathbf{u}^{2h} - \mathbf{v}^{2h}) \\
&= (cI_h^{2h} A^h I_{2h}^h(\mathbf{u}^{2h} - \mathbf{v}^{2h}), \mathbf{u}^{2h} - \mathbf{v}^{2h}) \\
&= (cA^{2h}(\mathbf{u}^{2h} - \mathbf{v}^{2h}), \mathbf{u}^{2h} - \mathbf{v}^{2h}) \\
&= c\|\mathbf{u}^{2h} - \mathbf{v}^{2h}\|_{A^{2h}}^2.
\end{aligned} \tag{6}$$

5.3 Discretization error

(a) Taylor expand $u(x)$ about x_i to approximate $u(x_{i-1})$ and $u(x_{i+1})$

$$\begin{aligned}
u(x_{i-1}) &= u(x_i) - u'(x_i)h + u''(x_i)\tfrac{h^2}{2} - u'''(x_i)\tfrac{h^3}{6} + u^{(iv)}(\xi^-)\tfrac{h^4}{24} \\
u(x_{i+1}) &= u(x_i) + u'(x_i)h + u''(x_i)\tfrac{h^2}{2} + u'''(x_i)\tfrac{h^3}{6} + u^{(iv)}(\xi^+)\tfrac{h^4}{24}
\end{aligned} \tag{7}$$

For $x_{i-1} < \xi^- < x_i$ and $x_i < \xi^+ < x_{i+1}$. Adding these two equations and solving for $-u''(x_i)$ gives

$$-u''(x_i) = \frac{-u(x_{i-1}) + 2u(x_i) - u(x_{i+1})}{h^2} + \frac{h^4}{24}(u^{(iv)}(\xi_-) + u^{(iv)}(\xi_+)). \tag{8}$$

Rearranging and writing in terms of $(A^h\mathbf{u})_i$, we have

$$(A^h\mathbf{u})_i = -u''(x_i) - \frac{h^2}{24}(u^{(iv)}(\xi^-) + u^{(iv)}(\xi^+)). \tag{9}$$

(b) If $u^{(iv)}$ is continuous, then by the Intermediate Value Theorem there must exist a ξ_i in $[\xi^-, \xi^+]$ such that

$$u^{(iv)}(\xi_i) = \frac{1}{2}(u^{(iv)}(\xi^-) + u^{(iv)}(\xi^+)). \tag{10}$$

Substituting this into the result from part (a) we have the truncation error at location x_i

$$\tau_i^h \equiv f(x_i) - (A^h\mathbf{u})_i = \frac{h^4}{24}(u^{(iv)}(\xi_-) + u^{(iv)}(\xi_+)) = \frac{h^2}{12}f''(\xi_i) \tag{11}$$

(c) First note that for any matrix B, the matrix h-norm is equivalent to the matrix l_2-norm. This is due to

$$\|B\|_h = \max_{\mathbf{x}\neq 0}\frac{\|B\mathbf{x}\|_h}{\|\mathbf{x}\|_h} = \max_{\mathbf{x}\neq 0}\frac{\sqrt{h}\|B\mathbf{x}\|_2}{\sqrt{h}\|\mathbf{x}\|_2} = \max_{\mathbf{x}\neq 0}\frac{\|B\mathbf{x}\|_2}{\|\mathbf{x}\|_2} = \|B\|_2. \tag{12}$$

Also, because $(A^h)^{-1}$ is symmetric,

$$\|(A^h)^{-1}\|_h = \max_i |\lambda_i[(A^h)^{-1}]| = \frac{1}{\min_i |\lambda_i[A^h]|} = \frac{h^2}{4\sin^2(h\pi/2)} \tag{13}$$

As h decreases monotonically from 1 to 0 this fraction decreases monotonically from $^1/_4$ to $^1/_{\pi^2}$. Therefore

$$\frac{1}{\pi^2} < \|(A^h)^{-1}\|_h < \frac{1}{4} \quad \text{for} \quad h \in (0,1). \tag{14}$$

Note that the second edition of the book has the bound stated differently, but I believe this is correct.

(d) Assume that $f''(x) < M$ for all $x \in [0,1]$.

$$\|\mathbf{v_i}\|_h = \sqrt{h\sum_{i=1}^{n-1} v_i^2} = \sqrt{h\sum_{i=1}^{n-1}[f''(\xi_i)]^2} \leq \sqrt{h\sum_{i=1}^{n-1} M^2} < \sqrt{hnM^2} = M. \tag{15}$$

This shows that $\|\mathbf{v_i}\|_h$ is bounded independent of problem size. We do not have the same result for the unscaled Euclidean norm

$$\|\mathbf{v_i}\|_2 = \sqrt{\sum_{i=1}^{n-1} v_i^2} = \sqrt{\sum_{i=1}^{n-1} [f''(\xi_i)]^2} \leq \sqrt{\sum_{i=1}^{n-1} M^2} = M\sqrt{n-1}. \tag{16}$$

As the problem size grows, so does the bound for $\|\mathbf{v_i}\|_2$.

(d) Under the many given assumptions, the discretization error can now be seen to be $\mathcal{O}(h^2)$

$$\begin{aligned} \|E^h\|_h &= \|\mathbf{u}^h - \mathbf{u}\|_h \\ &= \|(A^h)^{-1} A^h (\mathbf{u}^h - \mathbf{u})\|_h \\ &\leq \|(A^h)^{-1}\|_h \|A^h (\mathbf{u}^h - \mathbf{u})\|_h \\ &\leq \left(\tfrac{1}{4}\right)\left(\tfrac{h^2}{12} f''(\xi_i)\right) \\ &< \tfrac{M}{48} h^2 \end{aligned} \tag{17}$$

5.12 Two-dimensional five-point stencil. When the two-dimensional model problem is discretized on a uniform grid with $h_x = h_y = h$, the coefficients at each grid point are given by the five-point stencil

$$A^h = \frac{1}{h^2} \begin{pmatrix} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{pmatrix}. \tag{18}$$

What does the stencil for $A^{2h} = I_h^{2h} A^h I_{2h}^h$ look like if I_{2h}^h is based on bilinear interpolation and I_h^{2h} is based on (a) full weighting? (b) injection?

The stencil (of influence) of A^{2h} stands for a column of the coarse grid matrix. Therefore, consider applying $A^{2h} = I_h^{2h} A^h I_{2h}^h$ to the jth elementary basis vector $\hat{\mathbf{e}}_j$ in the coarse space given by a one at unknown i and zeros everywhere else. We will apply A^{2h} piece by piece. The stencil form for $\hat{\mathbf{e}}_j$, centered at point j and pictured only on local coarse gridpoints, is

$$\hat{\mathbf{e}}_j = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{19}$$

Interpolating with bilinear intepolation $\hat{\mathbf{e}}_j$ gives a stencil for local fine grid points

$$I_{2h}^h \hat{\mathbf{e}}_j = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{20}$$

Applying A^h to the interpolated stencil gives

$$A^h I_{2h}^h \hat{\mathbf{e}}_j = \frac{1}{h^2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{4} & -\frac{1}{2} & -\frac{1}{4} & 0 & 0 \\ 0 & -\frac{1}{4} & 0 & \frac{1}{2} & 0 & -\frac{1}{4} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 2 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{4} & 0 & \frac{1}{2} & 0 & -\frac{1}{4} & 0 \\ 0 & 0 & -\frac{1}{4} & -\frac{1}{2} & -\frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{21}$$

applying the full-weighting stencil to $A^h I_{2h}^h \hat{\mathbf{e}}_j$ and factoring out $\frac{1}{4}$ gives the answer to part (a)

$$A^h = I_h^{2h} A^h I_{2h}^h \hat{\mathbf{e}}_j = \frac{1}{(2h)^2} \begin{pmatrix} -\frac{1}{4} & -\frac{1}{2} & -\frac{1}{4} \\ -\frac{1}{2} & 3 & -\frac{1}{2} \\ -\frac{1}{4} & -\frac{1}{2} & -\frac{1}{4} \end{pmatrix}. \tag{22}$$

applying the injection stencil to $A^h I_{2h}^h \hat{\mathbf{e}}_j$ and factoring out $\frac{1}{4}$ gives the answer to part (b)

$$A^h = I_h^{2h} A^h I_{2h}^h \hat{\mathbf{e}}_j = \frac{1}{(2h)^2} \begin{pmatrix} 0 & -2 & 0 \\ -2 & 8 & -2 \\ 0 & -2 & 0 \end{pmatrix}. \tag{23}$$

It can be seen here that using injection for restriction will give us a coarse-grid operator of similar form to that of the fine grid.

5.13 Two-dimensional nine-point stencil. Repeat the previous problem with the nine-point stencil

$$A^h = \frac{1}{h^2}\begin{pmatrix} -1 & -1 & -1 \\ -1 & 8 & -1 \\ -1 & -1 & -1 \end{pmatrix}. \tag{24}$$

Again, the stencil (of influence) of A^{2h} stands for a column of the coarse grid matrix. Therefore, as in the previous problem, consider applying $A^{2h} = I_h^{2h} A^h I_{2h}^h$ to the jth elementary basis vector $\hat{\mathbf{e}}_j$ in the coarse space given by a one at unknown i and zeros everywhere else. We will apply A^{2h} piece by piece. The stencil form for $\hat{\mathbf{e}}_j$, centered at point j and pictured only on local coarse gridpoints, is

$$\hat{\mathbf{e}}_j = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{25}$$

Interpolating with bilinear intepolation $\hat{\mathbf{e}}_j$ gives a stencil for local fine grid points

$$I_{2h}^h \hat{\mathbf{e}}_j = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{26}$$

Applying A^h to the interpolated stencil gives

$$A^h I_{2h}^h \hat{\mathbf{e}}_j = \frac{1}{h^2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{4} & -\frac{3}{4} & -1 & -\frac{3}{4} & -\frac{1}{4} & 0 \\ 0 & -\frac{3}{4} & 0 & \frac{3}{2} & 0 & -\frac{3}{4} & 0 \\ 0 & -1 & \frac{3}{2} & 5 & \frac{3}{2} & -1 & 0 \\ 0 & -\frac{3}{4} & 0 & \frac{3}{2} & 0 & -\frac{3}{4} & 0 \\ 0 & -\frac{1}{4} & -\frac{3}{4} & -1 & -\frac{3}{4} & -\frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{27}$$

applying the full-weighting stencil to $A^h I_{2h}^h \hat{\mathbf{e}}_j$ and factoring out $\frac{1}{4}$ gives the answer to part (a)

$$A^h = I_h^{2h} A^h I_{2h}^h \hat{\mathbf{e}}_j = \frac{1}{(2h)^2} \begin{pmatrix} -1 & -1 & -1 \\ -1 & 8 & -1 \\ -1 & -1 & -1 \end{pmatrix}. \tag{28}$$

applying the injection stencil to $A^h I_{2h}^h \hat{\mathbf{e}}_j$ and factoring out $\frac{1}{4}$ gives the answer to part (b)

$$A^h = I_h^{2h} A^h I_{2h}^h \hat{\mathbf{e}}_j = \frac{1}{(2h)^2} \begin{pmatrix} -1 & -4 & -1 \\ -4 & 20 & -4 \\ -1 & -4 & -1 \end{pmatrix}. \tag{29}$$

It can be seen here that using full-weighting for restriction will give us a coarse-grid operator of similar form to that of the fine grid.