4.6 One-dimensional weighted Jacobi.

(a) Verify the Jacobi updating step (4.2).

Discretizing the equation $-u''(x) + c(x)u(x) = f(x)$ with centered differences gives the j equations

$$\frac{-v_{j-1} + 2v_j - v_{j+1}}{h^2} + c_j v_j = f_j. \tag{1}$$

Solving the jth equation for v_j yields the Jacobi update step

$$v_j^{m+1} = \frac{1}{2 + h^2 c_j}(v_{j-1}^m + v_{j+1}^m + h^2 f_j).$$

The weighted Jacobi update step is

$$v_j^{m+1} = \frac{\omega}{2 + h^2 c_j}(v_{j-1}^m + v_{j+1}^m + h^2 f_j) + (1-\omega)v_j^m.$$

(b) Show that the error $e_j = u_j - v_j$ satisfies (4.3).

It can easily be shown that the exact solution $\mathbf{u}$ is a fixed point for the weighted Jacobi step

$$u_j = \frac{\omega}{2 + h^2 c_j}(u_{j-1} + u_{j+1} + h^2 f_j) + (1-\omega)u^{j\cdot}$$

Then, the error update can be given by subtracting the update step for $\mathbf{v}$ from this equation

$$e_j^{m+1} = \frac{\omega}{2 + h^2 c_j}(e_{j-1}^m + e_{j+1}^m) + (1-\omega)e_j^m.$$

(c) Verify that the amplification factor for the method is given by

$$G(\theta) = 1 - 2\omega \sin^2\left(\frac{\theta}{2}\right). \tag{6}$$

Assume $c(x) = 0$. We try error components of a certain frequency θ in the error update equation. Assuming $e_j^m = A(m)e^{\iota j\theta}$, then plugging this in to the error update gives

$$\begin{aligned} A(m+1)e^{\iota j\theta} &= \tfrac{\omega}{2}(A(m)e^{\iota(j-1)\theta} + A(m)e^{\iota(j+1)\theta}) + (1-\omega)A(m)e^{\iota j\theta} \\ &= \tfrac{\omega}{2}(A(m)e^{\iota(j-1)\theta} + A(m)e^{\iota(j+1)\theta}) + (1-\omega)A(m)e^{\iota j\theta} \\ &= (\tfrac{\omega}{2}(e^{-\iota\theta} + e^{\iota\theta}) + 1 - \omega)A(m)e^{\iota j\theta} \\ &= (1 - \omega(1-\cos\theta))A(m)e^{\iota j\theta} \\ &= (1 - 2\omega\sin\tfrac{\theta}{2})A(m)e^{\iota j\theta} \end{aligned} \tag{7}$$

So the amplification factor is $G(\theta) = 1 - 2\omega \sin\frac{\theta}{2}$.

4.7 One-dimensional Gauss-Seidel. Verify that the error updating step (4.4). Then show that the amplification factor for the method is given by

$$G(\theta) = \frac{e^{\iota\theta}}{2 - e^{-\iota\theta}}. \tag{8}$$

Following the same argument for the last problem, the error update step for Gauss-Seidel is

$$e_j^{m+1} = \frac{1}{2 + h^2 c_j}(e_{j-1}^{m+1} + e_{j+1}^m). \tag{9}$$

Again, setting $c_j = 0$ and plugging in error of a certain frequency, we substitute $e_j^m = A(m)e^{\iota j\theta}$ into the error update step, and put all terms with $A(m+1)$ on the left-hand-side:

$$(2 - e^{-\iota\theta})A(m+1)e^{\iota j\theta} = e^{\iota\theta}A(m)e^{\iota j\theta}. \tag{10}$$

Solving for $A(m+1)$, we have

$$A(m+1) = \left(\frac{e^{\iota\theta}}{2 - e^{-\iota\theta}}\right)A(m) \tag{11}$$

4.11 Anisotropic operator. Consider the five-point stencil for the operator $-\epsilon u_{xx} - u_{yy}$ given by

$$\frac{1}{h^2}\begin{pmatrix} 0 & -1 & 0 \\ -\epsilon & 2(1+\epsilon) & -\epsilon \\ 0 & -1 & 0 \end{pmatrix}. \tag{12}$$

Find the amplification factors for weighted Jacobi and Gauss-Seidel applied to this system. Discuss the effect of the parameter ϵ in the case that $\epsilon << 1$.

For weighted-Jacobi the error update step is

$$e_{ij}^{m+1} = \frac{\omega}{2(1+\epsilon)}(\epsilon e_{(i-1),j}^m + e_{(i,(j-1)}^m + \epsilon e_{(i+1),j}^m + e_{i,(j+1)}^m) + (1-\omega)e_{ij}^m. \tag{13}$$

Now, to examine error that is a certain frequency θ_1 in the x-direction and another frequency θ_2 in the y-direction, we use $e_{ij}^m = A(m)e^{\iota(i\theta_1 + j\theta_2)}$. Then

$$\begin{aligned} A(m+1)e^{\iota(i\theta_1+j\theta_2)} &= A(m)e^{\iota(i\theta_1+j\theta_2)}\Big(\tfrac{\omega}{2(1+\epsilon)}(\epsilon e^{-\iota\theta_1} + e^{-\iota\theta_2} + ... \\ &\quad \epsilon e^{\iota\theta_1} + e^{\iota\theta_2}) + (1-\omega)\Big) \\ &= A(m)e^{\iota(i\theta_1+j\theta_2)}\Big(\tfrac{\omega}{(1+\epsilon)}(\cos(\theta_1) + \cos(\theta_2)... \\ &\quad -(1+\epsilon)) + 1\Big) \\ &= A(m)e^{\iota(i\theta_1+j\theta_2)}\Big(1 - \tfrac{2\omega}{1+\epsilon}(\epsilon \sin^2(\tfrac{\theta_1}{2}) + \sin^2(\tfrac{\theta_2}{2}))\Big). \end{aligned} \tag{14}$$

The amplication factor is therefore just

$$1 - \frac{2\omega}{1+\epsilon}(\epsilon \sin^2(\frac{\theta_1}{2}) + \sin^2(\frac{\theta_2}{2})). \tag{15}$$

As $\epsilon \to 0$, we see that this function is essentially the attenuation curve for the one-dimensional weighted-Jacobi error iteration in the y-direction for frequency θ_2.

For Gauss-Seidel, the error update step is

$$e_{ij}^{m+1} = \frac{1}{2(1+\epsilon)}(\epsilon e_{(i-1),j}^{m+1} + e_{(i,(j-1)}^{m+1} + \epsilon e_{(i+1),j}^m + e_{i,(j+1)}^m). \tag{16}$$

Now substitute $e_{ij}^m = A(m)e^{\iota(i\theta_1+j\theta_2)}$ and put all terms with $A(m+1)$ on the left-hand-side to get

$$A(m+1)e^{\iota(i\theta_1+j\theta_2)}\Big(2(1+\epsilon) - \epsilon e^{-\iota\theta_1} - e^{-\iota\theta_2}\Big) = A(m)e^{\iota(i\theta_1+j\theta_2)}\Big(\epsilon e^{\iota\theta_1} + e^{\iota\theta_2}\Big). \tag{17}$$

This yields the reduction factor

$$\frac{\epsilon e^{\iota\theta_1} + e^{\iota\theta_2}}{2(1+\epsilon) - \epsilon e^{-\iota\theta_1} - e^{-\iota\theta_2}}, \tag{18}$$

which again tends to the 1D case as $\epsilon \to 0$.