
3.2 An important equivalence. Consider a stationary, linear method of the form $\mathbf{v} \leftarrow \mathbf{v} + B^{-1}(\mathbf{f} - A\mathbf{v})$ applied to the problem $A\mathbf{u} = \mathbf{f}$. Use the following steps to show the relaxation on $Au = f$ with an arbitrary initial guess is equivalent to relaxation on $A\mathbf{e} = \mathbf{r}$ with the zero initial guess:

(a) First consider the problem $A\mathbf{u} = \mathbf{f}$ with arbitrary initial guess $\mathbf{v} = \mathbf{v}_0$. What are the error and residual associated with $\mathbf{v}_0$?

The associated error is $\mathbf{u} - \mathbf{v}_0$ and associated residual is $\mathbf{f} - A\mathbf{v}_0$. An iteration of the stationary, linear method is of the form $\mathbf{v}_1 \leftarrow \mathbf{v}_0 + B^{-1}\mathbf{r}_0$.

(b) Now consider the associated residual equation $A\mathbf{e} = \mathbf{r}_0 = \mathbf{f} - A\mathbf{v}_0$. What are the error and residual in the initial guess $\mathbf{e}_0 = \mathbf{0}$?

The solution to the equation $A\mathbf{e} = \mathbf{f} - A\mathbf{v}_0$ is $\mathbf{u} - \mathbf{v}_0$, so the associated error is $(\mathbf{u} - \mathbf{v}_0) - \mathbf{e}_0 = \mathbf{u} - \mathbf{v}_0$, and the associated residual is $\mathbf{r}_0 - A\mathbf{e}_0 = \mathbf{r}_0$. Here, an iteration of the stationary, linear method is of the form $\mathbf{e}_1 \leftarrow \mathbf{e}_0 + B^{-1}(\mathbf{r}_0 - A\mathbf{e}_0)$. Because $\mathbf{e}_0 = \mathbf{0}$, the iteration is actually $\mathbf{e}_1 \leftarrow \mathbf{e}_0 + B^{-1}\mathbf{r}_0$.

(c) Conclude that problems (a) and (b) are equivalent.

The probelms are equivalent in the sense that the associated error and residual equations are equivalent. The solution of the second system has just been *shifted* by the vector $\mathbf{v}_0$. Also, the itteration method would move both intial guesses by the same vector $B^{-1}\mathbf{r}_0$.

3.3 Properties of Interpolation. Show that I_{2h}^h based on linear interpolation is a linear operator with full rank in one and two dimensions.

Throughout this assignment, assume that the number of domain intervals in any axial direction is $n = 2^m$, where m is some integer. This sets the problem size per dimension at $(n-1)$, and grid spacing $h = 1/n$. This convention is not necessary but can avoid some painful book-keeping issues. In one dimension, linear interpolation is given on page 34 as

$$\begin{aligned} v_{2i}^h &= v_i^{2h} \\ v_{2i-1}^h &= \tfrac{1}{2}(v_{i-1}^{2h} + v_i^{2h}) \end{aligned} \tag{1}$$

This can be written as the following $(n-1) \times (\frac{n}{2}-1)$ matrix:

$$I_{2h}^h = \frac{1}{2}B := \frac{1}{2}\begin{bmatrix} 1 & & & \\ 2 & & & \\ 1 & 1 & & \\ & 2 & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \\ & & & 2 \\ & & & 1 \end{bmatrix}. \tag{2}$$

Note that the matrix B was defined so that this structured can be reused in the 2D case. Full rank can be shown by looking at the $(\frac{n}{2}-1)$ even-numbered row vectors and showing that they are linearly independent. Each of these row vectors is an elementary basis vector that is trivially independent from the other elementary basis vectors. The rank is thus at least $(\frac{n}{2}-1)$, but it is also at most $(\frac{n}{2}-1)$ because this is the minimal dimension of the matrix. Therefore, the 1D linear interpolation matrix has full rank.

As for the two-dimensional linear interpolation operator, the definition of bilinear interpolation is given on page 35 as

$$\begin{aligned} v_{2i,2j}^h &= v_{ij}^{2h} \\ v_{2i-1,2j}^h &= \tfrac{1}{2}(v_{i-1,j}^{2h} + v_i^{2h} j) \\ v_{2i,2j-1}^h &= \tfrac{1}{2}(v_{i,j-1}^{2h} + v_i^{2h} j) \\ v_{2i-1,2j-1}^h &= \tfrac{1}{4}(v_{i-1,j-1}^{2h} + v_{i,j-1}^{2h} + v_{i-1,j}^{2h} + v_i^{2h} j) \end{aligned} \tag{3}$$

We use blocks with the structure of the 1D interpolation matrix arrange to create the $(n-1)^2 \times (\frac{n}{2}-1)^2$ two-dimensional linear interpolation matrix:

$$I_{2h}^h = \frac{1}{4}\begin{bmatrix} B & & & \\ 2B & & & \\ B & B & & \\ & 2B & & \\ & B & & \\ & & \ddots & \\ & & & B \\ & & & 2B \\ & & & B \end{bmatrix}, \tag{4}$$

which is a constant multiple of the tensor product of B and itself. Regarding the rank, a similar argument can be made as the one made in the one-dimensional case. The $(\frac{n}{2}-1)$ even-numbered block-rows each contain only a $2B$ block and within each of these blocks are $(\frac{n}{2}-1)$ elementary basis row vectors and are linearly independent. These can be imbedded in and span $\mathbb{R}^{(\frac{n}{2}-1)^2}$. Therefore, the interpolation matrix has full rank. Following is some `MATLAB` code to create these interpolation matrices.

```
% number of intervals
n = 32;

% initialize empty matrix to store structure of 1D interpolation matrix
B = sparse(n-1,n/2-1);

% loop that creates the 1D structure
for j = 1:(n/2 -1)
B(2*j-1,j) = 1;
B(2*j, j) = 2;
B(2*j+1,j) = 1;
end

% 1D interpolation matrix
P1 = (1/2)*B;

% 2D interpolation matrix
P2 = (1/4)*kron(B, B);
```

3.4 Properties of Restriction. What is the rank of I_h^{2h} based on (a) full weighting and (b) injection in one and two dimensions?

In one dimension, the full weighting restriction operator is the following $(\frac{n}{2}-1)\times(n-1)$ matrix:

$$I_h^{2h} = \frac{1}{4}B^T = \frac{1}{4}\begin{bmatrix} 1 & 2 & 1 & & & & & & \\ & & 1 & 2 & 1 & & & & \\ & & & & & \ddots & & & \\ & & & & & & 1 & 2 & 1 \end{bmatrix}. \tag{5}$$

Note that this matrix is a constant multiple of the transpose of the linear interpolation matrix. To show that this matrix is full rank, we can use the same tactic as the previous problem. The only variation here is that we will look for linearly independent column vectors, instead of row vectors. The $(\frac{n}{2}-1)$ even-numbered column vectors are a constant times elementary basis vectors that span $\mathbb{R}^{(\frac{n}{2}-1)}$ and are trivially linearly independent.

Again, in one dimension, the injection restriction operator is the following $(\frac{n}{2}-1) \times (n-1)$ matrix:

$$I_h^{2h} = C^T := \begin{bmatrix} 0 & 1 & 0 & & & & & & \\ & & 0 & 1 & 0 & & & & \\ & & & & & \ddots & & & \\ & & & & & & 0 & 1 & 0 \end{bmatrix}. \tag{6}$$

Note that we defined C^T to be the one-dimensional injection matrix, as we will use it later in the two-dimensional case. Again, full rank is shown by noticing that the $(\frac{n}{2}-1)$ even-numbered column vectors are elementary basis vectors that span $\mathbb{R}^{(\frac{n}{2}-1)}$ and are trivially linearly independent.

For linear interpolation in two dimensions, we use the structure of the 1D interpolation given in B from the previous problem and form the two-dimensional linear restriction matrix:

$$I_h^{2h} = \frac{1}{16}\begin{bmatrix} B^T & 2B^T & B^T & & & & & & \\ & & B^T & 2B^T & B^T & & & & \\ & & & & & \ddots & & & \\ & & & & & & B^T & 2B^T & B^T \end{bmatrix}. \tag{7}$$

For injection, replace each block that has $2B^T$ in it with the matrix C^T from the one-dimensional injection restriction and replace each B^T block with the $(\frac{n}{2}-1)\times(n-1)$ matrix of all zeros.

To show that either of these matrices are full rank, use the same argument for the previous problem with column vectors instead of row vectors.

3.5 Null space of full weighting. Show that the null space of the full weighting operator, $N(I_h^{2h})$, has a basis consisting of vectors of the form

$$(0,0,...,-1,2,-1,...,0,0)^T. \tag{8}$$

By counting these vectors, show that the dimension of $N(I_h^{2h})$ is $\frac{n}{2}$.

Recall that we have required $n = 2^m$ to avoid book-keeping problems. The 1D full-weighting restriction operator I_h^{2h} is given in problem 3.4. The following vectors can all be easily verified to be in the null space of full-weighting restriction $\mathcal{N}(I_h^{2h})$:

$$\begin{array}{rcl} I_h^{2h}\begin{pmatrix} 2 & -1 & 0 & \dots & 0 \end{pmatrix}^T & = & \mathbf{0}, \text{ and} \\ I_h^{2h}\begin{pmatrix} 0 & \dots & 0 & -1 & 2 \end{pmatrix}^T & = & \mathbf{0}. \end{array} \tag{9}$$

Also, vectors of the following form are also easily seen to be in $\mathcal{N}(I_h^{2h})$:

$$I_h^{2h}\begin{pmatrix} 0 & \dots & -1 & 2 & -1 & \dots & 0 \end{pmatrix}^T = \mathbf{0}, \tag{10}$$

with the 2 centered at any odd interior point. Counting the two vectors at the end point plus the $\frac{n}{2} - 2$ vectors at odd interior points, we see that we have found $\frac{n}{2}$ vectors in $\mathcal{N}(I_h^{2h})$. The rank deficiency of I_h^{2h} is its largest dimension $(n-1)$ minus the rank $(\frac{n}{2} - 1)$, which is equal to $\frac{n}{2}$. The rank deficiency is the dimension of the null space. Because each of the $\frac{n}{2}$ null space vectors we found form a linearly independent set, we have a spanning basis of $\mathcal{N}(I_h^{2h})$.

3.6 Variational property.

(a) Let I_{2h}^h and I_h^{2h} be defined as in the text. Show that the linear interpolation operator and full weighting satisfy the variational property $I_{2h}^h = c(I_h^{2h})^T$ by computing $c \in \mathbf{R}$ for both one and two dimensions.

From problems 3.3 and 3.4, it can be easily seen what the constants are for the respective dimensions. For 1D, $c = 2$ and, for 2D, $c = 4$.

(b) The choice $c \neq 1$ found in part (a) is used because full weighting essentially preserves constants. Show that, except at the boundary, $I_h^{2h}(\mathbf{1}^h) = \mathbf{1}^{2h}$ (where $\mathbf{1}^h$ and $\mathbf{1}^{2h}$ are the vectors with entries 1 on their respective grids.)

The ith component of the vector $I_h^{2h}\mathbf{1}^h$ is the row sum of the matrix I_h^{2h}. Inspection of the one- and two-dimensional matrices in problem 3.4 reveals that row sum is equal to 1 for any point away from the boundary, or $(I_h^{2h}\mathbf{1}^h)_i = 1$ for any coarse-grid point i that isn't restricting information from the boundary. Thus, $I_h^{2h}(\mathbf{1}^h) = \mathbf{1}^{2h}$. Note that under the given problem and mesh spacing, we are never restricting information from the boundary. However, in general, boundary restriction does occur–for example, for odd interval number n or a problem with Neumann boundary conditions.
