PRECONDITIONING ITERATIVE METHODS FOR THE OPTIMAL CONTROL OF THE STOKES EQUATION

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Abstract. Solving problems regarding the optimal control of partial differential equations (PDEs) – also known as PDE-constrained optimization – is a frontier area of numerical analysis. Of particular interest is the problem of flow control, where one would like to effect some desired flow by exerting, for example, an external force. The bottleneck in many current algorithms is the solution of the optimality system – a system of equations in saddle point form that is usually very large and ill-conditioned. In this paper we describe a block-diagonal preconditioner for the minimal residual method which can be applied to such problems where the PDEs are the Stokes equations. We consider only distributed control here, although other problems – for example boundary control – can be treated in the same way. We give numerical results, and compare these with those obtained by solving the equivalent forward problem using similar techniques.

1. Introduction. Suppose that we have a flow that satisfies the Stokes equations in some domain Ω with some given boundary condition, and that we have some mechanism – for example, the application of a magnetic field – to change the forcing term on the right hand side of the PDE. Let $\widehat{\vec{v}}$ and $\hat{p}$ be given functions which are called the 'desired states'. Then the question is how do we choose the right hand side vector such that the velocity $\vec{v}$ and pressure p are as close as possible to $\widehat{\vec{v}}$ and $\hat{p}$, in some sense, while still satisfying the Stokes equations.

One way of formulating this problem is by minimizing a cost functional of tracking-type with the Stokes equations as a constraint, as follows:

$$\min_{v,p,u} \frac{1}{2}\|\vec{v}-\widehat{\vec{v}}\|^2_{L^2(\Omega)} + \frac{1}{2}\|p-\hat{p}\|^2_{L^2(\Omega)} + \frac{\beta}{2}\|\vec{u}\|^2_{L^2(\Omega)} \tag{1.1}$$

$$\begin{aligned} \text{s.t.} -\nabla^2\vec{v} + \nabla p &= \vec{u} \quad \text{in } \Omega \\ \nabla\cdot\vec{v} &= 0 \quad \text{in } \Omega, \\ \vec{v} &= \vec{w} \quad \text{on } \partial\Omega. \end{aligned}$$

Here $\vec{u}$ denotes the forcing term on the right hand side, which is known as the control. In order for the problem to be well-posed we also include the control in the cost functional, together with a Tikhonov regularization parameter β, which is usually chosen a priori.

There are two methods with which one can discretize this problem – we can either discretize the equations first and then optimize that system, or alternatively carry out the optimization first and then discretize the resulting optimality system. Since the Stokes equations are self-adjoint we will get the same discrete optimality system either way, provided the discretization methods are consistent between equations in the optimize-then-discretize technique. We will therefore only consider the discretize-then-optimize approach here.

Let $\{\vec{\phi}_j\},\ j=1,\dots,n_v+n_\partial$ and $\{\psi_k\},\ k=1,\dots,n_p$ be sets of finite element basis functions that form a stable discretization for the Stokes equations – see, for example, [5, Chapter 5] for further details – and define $\vec{v}_h = \sum_{i=1}^{n_v+n_\partial} V_i\vec{\phi}_i$ and $p_h = \sum_{i=1}^{n_p}$ be finite-dimensional approximations to $\vec{v}$ and p. Furthermore, let us also approximate the control from the velocity space, so $\vec{u}_h = \sum_{i=1}^{n_v} U_i\vec{\phi}_i$. The discrete Stokes equation

is of the form

$$\begin{bmatrix} \underline{K} & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \mathbf{p} \end{bmatrix} = Q_{\vec{v}}\mathbf{u} + \begin{bmatrix} \mathbf{f} \\ \mathbf{g} \end{bmatrix},$$

where $\mathbf{v}$, $\mathbf{p}$ and $\mathbf{u}$ are the coefficient vectors in the expansions of $\vec{v}_h$, p_h and $\vec{u}_h$ respectively, $\underline{K} = [\int_\Omega \nabla\vec{\phi}_i : \nabla\vec{\phi}_j]$, $B = [-\int_\Omega \psi_k \nabla\cdot\vec{\phi}_j]$, $Q_{\vec{v}} = [\int_\Omega \vec{\phi}_i\cdot\vec{\phi}_j]$, $\mathbf{f} = [-\sum_{j=n_u+1}^{n_u+n_\partial} U_j \int_\Omega \nabla\vec{\phi}_i : \nabla\vec{\phi}_j]$ and $\mathbf{g} = [\sum_{j=n_u+1}^{n_u+n_\partial} U_j \int_\Omega \psi_i \nabla\cdot\vec{\phi}_j]$.

On discretizing, the cost functional becomes

$$\min \frac{1}{2}\mathbf{v}^T Q_{\vec{v}}\mathbf{v} - \mathbf{v}^T\mathbf{b} + \frac{1}{2}\mathbf{p}^T Q_p \mathbf{p} - \mathbf{p}^T\mathbf{d} + \frac{\beta}{2}\mathbf{u}^T Q_{\vec{v}}\mathbf{u}$$

where $Q_p = [\int_\Omega \psi_i\psi_j]$, $\mathbf{b} = [\int_\Omega \widehat{\vec{v}}\vec{\phi}_i]$ and $\mathbf{d} = [\int_\Omega \widehat{p}\psi_i]$.

Let us introduce two vectors of Lagrange multipliers, $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$. Then minimizing the Lagrangian function gives the discrete optimality system of the form

$$\begin{bmatrix} Q_{\vec{v}} & 0 & 0 & \underline{K} & B^T \\ 0 & Q_p & 0 & B & 0 \\ 0 & 0 & \beta Q_{\vec{v}} & -Q_{\vec{v}}^T & 0 \\ \underline{K} & B^T & -Q_{\vec{v}} & 0 & 0 \\ B & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \mathbf{p} \\ \mathbf{u} \\ \boldsymbol{\lambda} \\ \boldsymbol{\mu} \end{bmatrix} = \begin{bmatrix} \mathbf{b} \\ \mathbf{d} \\ \mathbf{0} \\ \mathbf{f} \\ \mathbf{g} \end{bmatrix}. \tag{1.2}$$

It will be useful to relabel this system so that it becomes

$$\begin{bmatrix} \mathcal{Q} & 0 & \mathcal{K} \\ 0 & \beta Q_{\vec{v}} & -\widehat{Q}^T \\ \mathcal{K} & -\widehat{Q} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \mathbf{u} \\ \boldsymbol{\xi} \end{bmatrix} = \begin{bmatrix} \mathbf{c} \\ \mathbf{0} \\ \mathbf{h} \end{bmatrix}, \tag{1.3}$$

where $\mathcal{Q} = \texttt{blkdiag}(Q_{\vec{v}}, Q_p)$, $\mathcal{K} = \begin{bmatrix} \underline{K} & B^T \\ B & 0 \end{bmatrix}$, $\widehat{Q} = [Q_{\vec{v}}\ 0]^T$ and the vectors $\mathbf{y}$, $\boldsymbol{\xi}$, $\mathbf{c}$ and $\mathbf{h}$ take their obvious definitions. For more detail on the practicalities of discretizing control problems of this type, see Rees, Stoll and Wathen [13]. Finding an efficient method to solve this system will be the topic of the remainder of the paper.

The matrix in (1.3) is of saddle point form, that is

$$\mathcal{A} = \begin{bmatrix} A & C^T \\ C & 0 \end{bmatrix},$$

where $A = \texttt{blkdiag}(\mathcal{Q}, \beta Q_{\vec{v}})$ and $C = [\mathcal{K}\ \ -\widehat{Q}]$. The matrix $\mathcal{A}$ is, in general, very large – just one component of it is the discrete Stokes equations – yet is sparse. A good choice for solving such systems are iterative methods – in particular Krylov subspace methods. It is well known that matrices of the form $\mathcal{A}$ are indefinite, and the method of choice for such systems is the minimal residual method (MINRES) of Paige and Saunders [10].

For MINRES to be efficient for such a matrix we need to combine the method with a good preconditioner – i.e. a matrix $\mathcal{P}$ which is cheap to invert and which clusters the eigenvalues of $\mathcal{P}^{-1}\mathcal{A}$. One method that is often used is to look for a block diagonal preconditioner of the form

$$\mathcal{P} = \begin{bmatrix} A_0 & 0 \\ 0 & S_0 \end{bmatrix}.$$

It is well known (see, for example, [5, Theorem 6.6]) that if A, A_0, $CA^{-1}C^T$ and S_0 are positive definite matrices such that there exist constants δ, Δ, ϕ and Φ such that the generalized Rayleigh quotients satisfy

$$\delta \leq \frac{\mathbf{x}^T A \mathbf{x}}{\mathbf{x}^T A_0 \mathbf{x}} \leq \Delta, \quad \phi \leq \frac{\mathbf{y}^T C A^{-1} C^T \mathbf{y}}{\mathbf{y}^T S_0 \mathbf{y}} \leq \Phi$$

for all vectors $\mathbf{x} \in \mathbb{R}^{2n_v+n_p}$ and $\mathbf{y} \in \mathbb{R}^{n_v+n_p}$, $\mathbf{x},\ \mathbf{y} \neq \mathbf{0}$, then the eigenvalues λ of $\mathcal{P}^{-1}\mathcal{A}$ are real, and satisfy

$$\frac{\delta - \sqrt{\delta^2 + 4\Delta\Phi}}{2} \leq \lambda \leq \frac{\Delta - \sqrt{\Delta^2 + 4\phi\delta}}{2},$$
$$\delta \leq \lambda \leq \Delta,$$
or
$$\frac{\delta + \sqrt{\delta^2 + 4\delta\phi}}{2} \leq \lambda \leq \frac{\Delta + \sqrt{\Delta^2 + 4\Phi\Delta}}{2}.$$

Therefore, if we can find matrices A_0 and S_0 that are cheap to invert and are good approximations to A and the Schur complement $CA^{-1}C^T$ in the sense defined above, then we will have a good preconditioner, since the eigenvalues of $\mathcal{P}^{-1}\mathcal{A}$ will be in three distinct clusters bounded away from 0. In the ideal case where $A_0 = A$ and $S_0 = CA^{-1}C^T$ we have $\delta = \Delta = \phi = \Phi = 1$. Then the preconditioned system will have precisely three eigenvalues, 1, $\frac{1+\sqrt{5}}{2}$ and $\frac{1-\sqrt{5}}{2}$, so MINRES would converge in three iterations [9].

2. Choosing A_0. Suppose, for simplicity, that our domain $\Omega \subset \mathbb{R}^2$. If, as is usual, we use the same element space for all components in the velocity vector, and this has basis $\{\phi_i\}$. Then $Q_{\vec{v}} = \texttt{blkdiag}(Q_v, Q_v)$, where $Q_v = [\int_\Omega \phi_i \phi_j]$. Then the matrix A is just a block diagonal matrix composed of the mass matrices in the bases $\{\phi_i\}$ or $\{\psi_i\}$. Wathen [16] showed that for a general mass matrix, Q, if $D := \texttt{diag}(Q)$, then it is possible to calculate constants ξ and Ξ such that

$$\xi \leq \lambda(D^{-1}Q) \leq \Xi.$$

The constants depend on the elements used – for example, for $\mathbf{Q}_1$ elements $\xi = 1/4,\ \Xi = 9/4$ and for $\mathbf{Q}_2$ elements $\xi = 0.3103,\ \Xi = 1.5625$. The diagonal would therefore be a good approximation to A.

However, as A is in a sense 'easy' to invert, it would help to have as good an approximation here as we can. Using the bounds described above we have all the information we need to use the relaxed Jacobi method accelerated by the Chebyshev-semi iteration. This is a method that is very cheap to use and, as demonstrated by Wathen and Rees in [17], is particularly effective in this case. In particular, since the eigenvalues of $D^{-1}M$ are evenly distributed, there is very little difference between the convergence of this method and the non-linear conjugate gradient method [7] preconditioned with D. Note that since the conjugate gradient algorithm is non-linear, we cannot use it as a preconditioner for a stationary Krylov subspace method such as MINRES, unless run to convergence. The Chebyshev semi-iteration, on the other hand, is a linear method. Suppose we use it to solve $Q\mathbf{x} = \mathbf{b}$ for some right hand side $\mathbf{b}$. Then we can write every iteration as $\mathbf{x}^{(m)} = T_m^{-1}\mathbf{b}$, for some matrix T_m implicitly defined by the method which is independent of $\mathbf{b}$.

By choosing a larger m, T_m gets to be a better approximation to Q. Table I in Rees and Stoll [12] gives the upper and lower bounds for each m from 1 to 20 for a $\mathbf{Q}_1$

discretization. Therefore, if $\delta_m^v \leq \lambda\left(\left(T_m^v\right)^{-1} Q_v\right) \leq \Delta_m^v$ and $\delta_m^p \leq \lambda\left(\left(T_m^p\right)^{-1} Q_v\right) \leq \Delta_m^p$, then

$$\delta_m \leq \frac{\mathbf{x}^T A \mathbf{x}}{\mathbf{x}^T A_0 \mathbf{x}} \leq \Delta_m, \tag{2.1}$$

where $A_0 = \texttt{blkdiag}(T_m^v, T_m^v, T_m^p, \beta T_m^v, \beta T_m^v)$ and $\delta_m = \min(\delta_m^v, \delta_m^p)$ and $\Delta_m = \max(\Delta_m^v, \Delta_m^p)$, both independent of the mesh size, h. We therefore have an inexpensive way to make the bounds on $\lambda(A_0^{-1}A)$ as close to unity as needed.

3. Choosing S_0. Now consider the Schur complement, $S := \frac{1}{\beta}\widehat{Q}Q_{\vec{v}}^{-1}\widehat{Q}^T + \mathcal{K}\mathcal{Q}^{-1}\mathcal{K}$. The dominant term in this sum is, for all but the smallest values of β, $\mathcal{K}\mathcal{Q}^{-1}\mathcal{K}$ – the term that contains the PDE. Figure 3.1 shows the eigenvalue distribution for this approximation of S for a relatively coarse $\mathbf{Q}_2 - \mathbf{Q}_1$ discretization with $\beta = 0.01$. As we can see from the figure, this clusters the eigenvalues nicely, and so we can expect good convergence of MINRES if we used this as S_0.

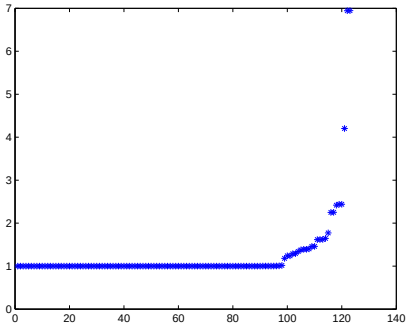


FIG. 3.1. *Eigenvalues of* $(\mathcal{K}\mathcal{Q}^{-1}\mathcal{K})^{-1}S$

However, a preconditioner must be easy to invert, and solving a system with $\mathcal{K}\mathcal{Q}^{-1}\mathcal{K}$ requires two solves with the discrete Stokes matrix, which is not cheap. We therefore would like some matrix, $\widetilde{\mathcal{K}}$, such that $\widetilde{\mathcal{K}}\mathcal{Q}^{-1}\widetilde{\mathcal{K}}$ approximates $\mathcal{K}\mathcal{Q}^{-1}\mathcal{K}$. Note that the mass matrices are not important here and it is sufficient that $\widetilde{\mathcal{K}}\widetilde{\mathcal{K}}^T$ approximates $\mathcal{K}^2$.

Braess and Peisker [2] show that it is not sufficient that $\widetilde{\mathcal{K}}$ approximates $\mathcal{K}$. Indeed, for the Stokes equation Silvester and Wathen [14] showed that an ideal preconditioner is $\texttt{blkdiag}(\underline{K}, M_p)$, where $\underline{K}$ is a multigrid cycle, but the eigenvalues of $(\widehat{\mathcal{K}}\widehat{\mathcal{K}}^T)^{-1}\mathcal{K}^2$ are not at all clustered, and the approximation is a poor one in this case. Suppose we wish to solve the equation $\mathcal{K}\mathbf{x} = \mathbf{b}$, for some right hand side vector $\mathbf{b}$. Braess and Peisker go on to show that if we take an approximation $\mathcal{K}_m$ which is implicitly defined by an iteration such that $\mathbf{x}^{(m)} = \mathcal{K}_m^{-1}\mathbf{b}$, say, which converges to the solution $\mathbf{x}$ in the sense that

$$\|\mathbf{x}^{(m)} - \mathbf{x}\| \leq \eta_m \|\mathbf{x}\|,$$

then $\eta_m = \|\mathcal{K}_m^{-1}\mathcal{K} - I\|$, and one can show

$$(1-\eta)^2 \leq \frac{\mathbf{x}^T\mathcal{K}^2\mathbf{x}}{\mathbf{x}^T\mathcal{K}_m^T\mathcal{K}_m\mathbf{x}} \leq (1+\eta)^2.$$

Hence, such an approximation is suitable for use in this case.

Note that MINRES cannot be used to approximate $\mathcal{K}$, unless run until convergence, since – like CG – MINRES is a Krylov subspace method, and hence nonlinear. We would therefore have to use a flexible outer method if we were to make this approximation. As before, consider a simple iteration of the form

$$\mathbf{x}^{(m+1)} = \mathbf{x}^{(m)} + M^{-1}\mathcal{K}\mathbf{r}^{(m)},$$

where $\mathbf{r}^{(m)}$ is the residual at the m^{th} step, and with a block lower-triangular splitting matrix

$$M := \begin{bmatrix} \underline{K}_0 & 0 \\ B & -Q_0 \end{bmatrix},$$

where $\underline{K}_0$ approximates $\underline{K}$ and Q_0 approximates Q_p, which is itself spectrally equivalent to the Schur complement [5, Section 6.2]. By the result of Braess and Peisker, we just need to show that this iteration converges – i.e. that $\rho(I - M^{-1}\mathcal{K}) < 1$, where ρ denotes the spectral radius – to get that this defines a good approximation to the square. We ignore the one zero eigenvalue of $\underline{K}$ which is due to the hydrostatic pressure here, and in what follows, since if we start an iteration orthogonal to this kernel, we will remain orthogonal to the kernel [5, Section 2.3].

Consider two cases – $\underline{K} - \underline{K}_0$ positive definite, and $\underline{K} - \underline{K}_0$ indefinite. In the first case, Theorem 3.1 in Rees and Stoll [12] shows that if $\underline{K}_0$ and Q_0 are positive definite matrices such that

$$\upsilon \leq \frac{\mathbf{x}^T\underline{K}\mathbf{x}}{\mathbf{x}^T\underline{K}_0\mathbf{x}} \leq \Upsilon, \quad \psi \leq \frac{\mathbf{y}^T B\underline{K}^{-1}B^T\mathbf{y}}{\mathbf{y}^T Q_0\mathbf{y}} \leq \Psi,$$

then λ is real and positive, and moreover satisfies

$$\begin{aligned} \frac{(1+\psi)\Upsilon - \sqrt{(1+\psi)^2\Upsilon^2 - 4\psi\Upsilon}}{2} \leq &\lambda \leq \frac{(1+\Psi)\upsilon - \sqrt{(1+\Psi)^2\upsilon^2 - 4\Psi\upsilon}}{2} \\ \upsilon \leq &\lambda \leq \Upsilon \qquad \text{or} \\ \frac{(1+\psi)\upsilon + \sqrt{(1+\psi)^2\upsilon^2 - 4\psi\upsilon}}{2} \leq &\lambda \leq \frac{(1+\Psi)\Upsilon + \sqrt{(1+\Psi)^2\Upsilon^2 - 4\Psi\Upsilon}}{2}. \end{aligned}$$

We would like to put some numbers to these bounds in order to see what this means for a simple iteration based on a splitting with the block lower triangular matrix above. It is well known that a multigrid iteration is a good approximation to $\underline{K}$, and we can scale such an iteration so that

$$1 \leq \frac{\mathbf{x}^T\underline{K}\mathbf{x}}{\mathbf{x}\underline{K}_0\mathbf{x}} \leq \frac{1+\rho^m}{1-\rho^m},$$

where m is the number of V-cycles. A realistic value for ρ is 0.15 (see [5, pp. 294-295]), and experimentation shows $m = 2$ gives reasonable performance. Using Q_p for the Schur complement approximation we have

$$\gamma^2 \leq \frac{\mathbf{x}^T B\underline{K}^{-1}B^T\mathbf{x}}{\mathbf{x}^T Q_p\mathbf{x}} \leq \Gamma^2,$$

$\mathbf{x} \neq \mathbf{1}$, where for 2D Q1 elements, $\gamma^2 = 0.2$, $\Gamma^2 = 1$. Approximating this by 10 steps of the Chebyshev semi-iteration will weaken these bounds by a factor of 0.96

in the lower bound and $\Theta = 1.04$ in the upper. With these numbers, we have that $\lambda(P_{BLT}^{-1}\mathcal{A}) \in [0.19, 1.29]$, and hence $\rho(I - P_{BLT}^{-1}\mathcal{A}) = 0.81 < 1$. Therefore a simple iteration with this splitting will converge.

Although we've assumed $\underline{K} - \underline{K}_0 \geq 0$ in the analysis above, experiments show we still have good convergence properties even if this isn't true. Extending the result above to the case where $\underline{K} - \underline{K}_0$ is indefinite, it can be shown that the real eigenvalues of $M^{-1}\mathcal{K}$ satisfy

$$\begin{aligned} \frac{(1+\psi)\Upsilon - \sqrt{(1+\psi)^2\Upsilon^2 - 4\psi\Upsilon}}{2} \leq &\lambda \leq \frac{(1+\Psi)\Upsilon + \sqrt{(1+\Psi)^2\Upsilon^2 - 4\Psi\Upsilon}}{2} \\ \text{or} \qquad \upsilon \leq &\lambda \leq \Upsilon, \end{aligned}$$

and the complex eigenvalues can be written as $\lambda = re^{i\theta}$, where r and θ satisfy

$$\sqrt{\upsilon\psi} \leq r \leq \sqrt{\Psi}, \ -\tan^{-1}(\sqrt{\upsilon^{-1} - 1}) \leq \theta \leq \tan^{-1}(\sqrt{\upsilon^{-1} - 1}).$$

A simple geometrical argument will show that if we define

$$\begin{aligned} \sigma := \max \Bigg\{ & 1 - \upsilon, \Upsilon - 1, 1 - \frac{(1+\psi)\Upsilon - \sqrt{(1+\psi)^2\Upsilon^2 - 4\psi\Upsilon}}{2}, \\ & \frac{(1+\Psi)\Upsilon + \sqrt{(1+\Psi)^2\Upsilon^2 - 4\Psi\Upsilon}}{2} - 1, \sqrt{1 + \Psi - 2\sqrt{\Psi}\cos\theta}, \\ & \sqrt{1 + \psi\upsilon - 2\sqrt{\psi\upsilon}\cos\theta} \Bigg\}, \end{aligned}$$

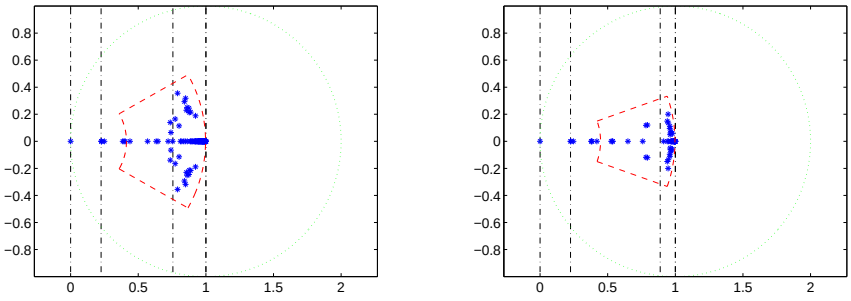
then the simple iteration based on the splitting matrix M will converge if $\sigma < 1$, with the asymptotic convergence rate being σ. Figure 3.2 shows the bounds predicted above and the actual eigenvalues for a number of approximations to the matrix $\underline{K}$.

These show asymptotic convergence, but in practice we see good results from the first iteration. In fact, experimentation shows that if you want to spend m multigrid cycles, say, per iteration, then it is most beneficial to use approximate $\underline{K}$ with all m cycles, and only do one iteration. Such good convergence may be explained, at least in the case where $\underline{K} - \underline{K}_0 > 0$, by the fact that $M^{-1}\mathcal{K}$ is self-adjoint in the inner product defined by $\texttt{blkdiag}(\underline{K} - \underline{K}_0, Q_0)$ (see, for example, Bramble and Pasciak [1] for more details). This means that the $M^{-1}\mathcal{K}$ is normal, and so convergence, when measured in the relevant norm, will be monotonic.

We therefore advocate using $S_0 = M\mathcal{Q}^{-1}M^T$ as an approximation to the Schur complement, where $\underline{K}$ is a good approximation to $\underline{K}$ – say four V-cycles – and Q_0 is given by a number of steps of the Chebyshev semi-iteration. A matrix of the form

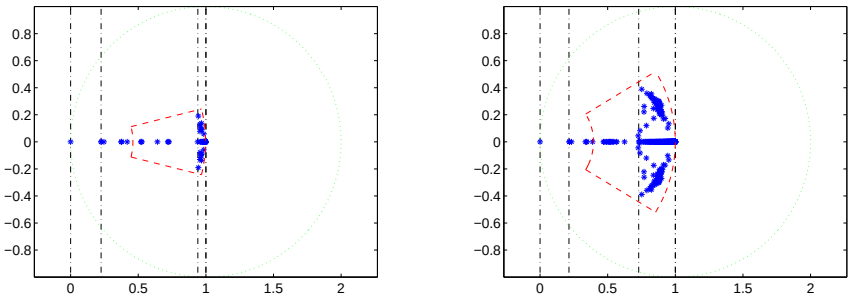
$$\mathcal{P} := \begin{bmatrix} A_0 & 0 \\ 0 & M\mathcal{Q}^{-1}M^T \end{bmatrix},$$

where A_0 is composed of Chebyshev approximations, should therefore be an effective preconditioner for the matrix $\mathcal{A}$.



(a) h = 0.25, K_0 given by 1 AMG V-cycle with 1 pre- and 1 post-smoothing step

(b) h = 0.25, K_0 given by 1 AMG V-cycle with 2 pre- and 2 post-smoothing steps



(c) h = 0.25, K_0 given by 2 AMG V-cycles with 2 pre- and 2 post-smoothing steps

(d) h = 0.125, K_0 given by 1 AMG V-cycle with 1 pre- and 1 post-smoothing step

FIG. 3.2. **'s denote computed eigenvalues. Lines, from left to right, are at 0, $\frac{(\psi+1)\Upsilon-\sqrt{(\psi+1)^2\Upsilon^2-4\psi\Upsilon}}{2}$, υ, Υ and $\frac{(\Psi+1)\Upsilon+\sqrt{(\Psi+1)^2\Upsilon^2-4\Psi\Upsilon}}{2}$. (the last two virtually coincide here). Dashed region is the bounds for the complex eigenvalues. Also shown is the unit circle centred at $z = 1$.*

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4. Numerical Results. First, consider the following forward problem, which sets the boundary conditions that we will use for the control problem. This is a classic test problem in fluid dynamics called leaky cavity flow, and a discussion is given by Elman, Silvester and Wathen [5, Example 5.1.3].

EXAMPLE 4.1. *Let $\Omega = [0,1]^2$, and let $\mathbf{i}$ and $\mathbf{j}$ denote unit vectors in the direction of the x and y axis respectively. Let $\vec{v}$ and p satisfy the Stokes equations*

$$\begin{aligned} -\nabla^2\vec{v} + \nabla p &= \vec{0} \quad \text{in } \Omega \\ \nabla\cdot\vec{v} &= 0 \quad \text{in } \Omega, \end{aligned}$$

and let $\vec{v} = \vec{0}$ on the boundary, except for on $x = 1$, where $\vec{u} = -\mathbf{j}$.

We discretize this using $\mathbf{Q}_2 - \mathbf{Q}_1$ elements and solve the resulting system using MINRES. As a preconditioner we use the block diagonal matrix $\texttt{blkdiag}(\widehat{K}, T_{20})$, following Silvester and Wathen [14], where $\widehat{K}$ denotes one AMG V-cycle (using HSL MI20 [3] applied via a MATLAB interface) and T_{20}^{-1} is twenty steps of the Chebyshev semi-iteration applied with the pressure mass matrix. The problem was solved using MATLAB R2009b, and the number of iterations and the time taken for different mesh

sizes is given in Table 4.1.

h	Iterations	CPU time (s)
2^{-2} (187)	19	0.015
2^{-3} (659)	24	0.073
2^{-4} (2,467)	26	0.082
2^{-5} (9,539)	28	0.21
2^{-6} (37,507)	29	3.80
2^{-7} (148,739)	29	15.5

TABLE 4.1
Number of MINRES iterations and time taken to solve the forward problem in Example 4.1

Figure 4.1 shows the streamlines and the pressure of the solution obtained. Note the small recirculations present in the lower corners – called Moffatt eddies. Adding a forcing term that reduces these eddies will be the object of our control problem, Example 4.2.

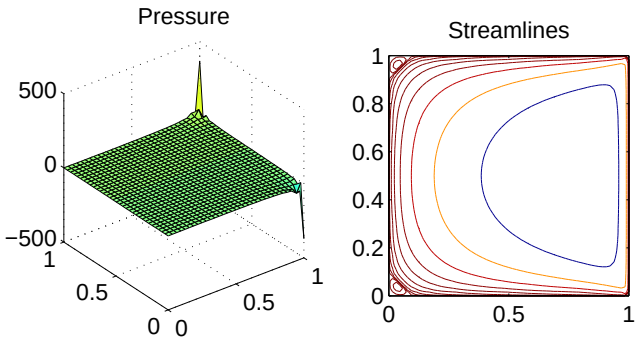

FIG. 4.1. *Solution of Example 4.1*

EXAMPLE 4.2. *Let $\Omega = [0,1]^2$ and consider an optimal control problem of the form (1.1), with Dirichlet boundary conditions as given in Example 4.1. Take the desired pressure as $\widehat{p} = 0$ and let $\widehat{v}$ be as in Figure 4.2.*

We took the regularization parameter as $\beta = 0.01$ – a value regularly used in the literature [4, 6, 8]. As above, we discretize this using $\mathbf{Q}_2 - \mathbf{Q}_1$ elements – and hence use $\mathbf{Q}_2$ elements for the control – and solve the resulting system using MINRES. As a preconditioner we use the block diagonal matrix $\mathcal{P}$. We take four AMG V-cycles (again using HSL MI20) for $\underline{K}_0$ and we use twenty steps of the Chebyshev semi-iteration applied in place of a mass matrix. The problem was solved using MATLAB R2009b, and the number of iterations and the time taken for different mesh sizes is given in Table 4.1.

Comparing the results in Table 4.2 and Table 4.1, we see that in both cases the iteration numbers do not increase significantly with the mesh size, and also that the

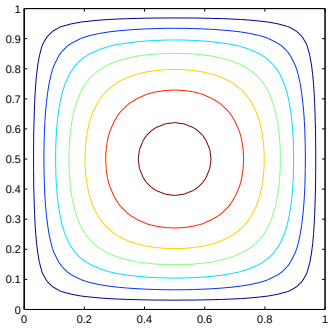

FIG. 4.2. *The desired velocity, $\widehat{v}$, in Example 4.2*

h	Iterations	CPU time (s)
2^{-2} (344)	26	0.48
2^{-3} (1,512)	31	1.05
2^{-4} (6,344)	33	3.69
2^{-5} (25,992)	33	18.0
2^{-6} (105,224)	34	84.2
2^{-7} (423,432)	34	342

TABLE 4.2
Number of MINRES iterations and time taken to solve the control problem in Example 4.2

times scale linearly as the problem size increases. The control problem takes longer to solve, as one might expect, but the number of iterations taken to solve this is only a handful more than the number required to solve the forward problem. Looking at the times taken to solve the problems on the same grid it appears that a solve of the control problem is about 22 times more expensive than a solve of the forward problem – a overhead that seems reasonable, given the increased complexity of the control problem.

We have only presented a simple distributed control problem here. It is possible to solve other types of control problem using the same method – see [11] for a discussion in the simpler case of Poisson control. It is also possible to use this method together with bound constraints on the control – Stoll and Wathen [15] discuss this approach in consideration of the Poisson control problem.

5. Conclusions. In this paper we have presented a preconditioner that can be used to solve problems in Stokes control. We have given some theoretical justification for the effectiveness of such a preconditioner and have given some numerical results. We compared these results with those for solving the equivalent forward problem, and the iteration count is only marginally higher in the control case, and behaves in broadly the same way as the iterations taken the solve the forward problem as the mesh size decreases. This preconditioner therefore seems reasonable for problems of this type. Furthermore, the ideas presented here have the potential to be extended to develop preconditioners for a variety of problems, with the additional constraints and features that real-world applications require.

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