OPTIMAL MULTILEVEL AND ADAPTIVE FINITE ELEMENT METHODS FOR TIME-HARMONIC MAXWELL EQUATIONS

LIUQIANG ZHONG
SCHOOL OF MATHEMATICS AND COMPUTATIONAL SCIENCES, XIANGTAN UNIVERSITY, CHINA

ABSTRACT. Three related results in this paper will be presented for the finite element discretization of time-harmonic Maxwell equations: a two-grid method, L^2 error estimates and convergence and complexity of adaptive finite element methods. Numerical experiments are carried out to justify the optimality of these results.

1. INTRODUCTION

In this paper, we shall study the classical time-harmonic Maxwell equations in three dimensions of the following form (with moderate size of frequency ω):

$$\nabla \times (\mu^{-1}\nabla \times \boldsymbol{E}) - \omega^2(\epsilon + i\sigma/\omega)\boldsymbol{E} = \boldsymbol{F} \quad \text{in } \Omega, \tag{1.1}$$

$$\boldsymbol{\nu} \times \boldsymbol{E} = \boldsymbol{0} \quad \text{on } \partial\Omega. \tag{1.2}$$

We will be interested in the basic algorithmic development and analysis of the finite element discretizations and the corresponding multilevel and adaptive methods for the above model equations.

We firstly present a two-grid method for the Nédélec's second type element discretizations. The two-grid method, originally developed by Xu in early 90s for elliptic boundary value problems, transforms an indefinite problem discretized on a fine grid into a symmetric positive definite (SPD) problem on the same grid together with a discretization of the original indefinite problem on a much coarse grid. Because of the leading term in (1.1)-(1.2) has a large kernel, the extension of the two grid method to (1.1)-(1.2) is not straightforward. Thanks to an optimal L^2 error estimate which will be introduced in §2.2, we are able to obtain an optimal two-grid result for (1.1)-(1.2) in §2.1.

Secondly, we present an optimal L^2 error estimate for the Nédélec's type element discretizations. Note to mention that this type of estimate is of fundamental importance in finite element theory, it is also instrumental in developing and analyzing multilevel and adaptive finite elements. However, one very interesting observation we have made is that an optimal L^2 error estimate has still not yet rigorously established in the literature. Despite of the much effort given in the literature, we found either the estimate is not optimal, see Monk [6], or the proof has a nontrivial gap, see Hiptmair [3]. In this paper, we will settle this issue and present an optimal L^2 estimate in §2.2.

Finally, we concern the optimality of adaptive finite element methods (AFEM), which is a result of extensive research interests, see Cascon *et. al.* [1] as one most recent example of work for second order elliptic boundary value problems. The best resutl for $\boldsymbol{H}(\mathbf{curl})$ positive definite problems as far as we know, is Hoppe and Schoberl [5]. They proved the convergence of adaptive edge finite element methods (AEFEM) for the three dimensional cases with the so-called interior node property and extra marking for oscillation. With the techniques in Cascon *et. al.* [1] and several other techniques we developed (including the aforementioned L^2 error estimates), we are able to establish optimal estimates, for both error and complexity, on AEFEM for (1.1)-(1.2) without using interior node property and extra marking for oscillation, see §2.3.

We have also carried out extensive numerical computations to implement various algorithms in study and to verify the estimates that we have obtained. We will report these numerical results in §2.1 and §2.3, respectively.

To avoid the repeated use of generic but unspecified constants, we use the notation $a \lesssim b$ means that there exists a positive constant C such that $a \leq Cb$, the above generic constants C are independent of the function under consideration, but they may depend on parameters of model and the shape-regularity of the meshes.

2. Main results

We first describe some details for the model problem (1.1)-(1.2) and the finite element method to discretize it. We assume that Ω is a Lipschitz polyhedron in $\mathbb{R}^3$ with connected boundary $\partial\Omega$ and unit outwards normal $\boldsymbol{\nu}$, μ is the magnetic permeability, $\omega > 0$ is the angular frequency, $i = \sqrt{-1}$, ϵ and σ are the electric permittivity and conductivity of the homogeneous isotropic body occupying Ω, and $\boldsymbol{F} = i\omega\boldsymbol{J}$ with the applied current density $\boldsymbol{J}$. In general, μ and ϵ are positive definite tensor functions, and σ is positive definite in a conductor and vanishes in an insulator. In particular, (1.1)-(1.2) describe the eddy current model when $\epsilon = 0$ and the lossless case of the indefinite time-harmonic Maxwell equations when $\sigma = 0$. For simplicity of exposition, we assume that $\mu = 1$, both ϵ and σ are positive constants, and $\boldsymbol{F} \in (L^2(\Omega))^3$.

Let us introduce the Sobolev space $H_0(\mathbf{curl};\Omega) = \{\boldsymbol{u} \in (L^2(\Omega))^3, \nabla\times\boldsymbol{u} \in (L^2(\Omega))^3, (\boldsymbol{\nu}\times\boldsymbol{u})|_{\partial\Omega} = \mathbf{0}\}$ associated with the norm $\|\boldsymbol{v}\|_{H(\mathbf{curl};\Omega)} := (\|\nabla\times\boldsymbol{u}\|_0^2 + \|\boldsymbol{u}\|_0^2)^{1/2}$, where $\|\cdot\|_0$ denotes the norm in $(L^2(\Omega))^3$. The variational formulation of equations (1.1)-(1.2) is: Find $\boldsymbol{E} \in H_0(\mathbf{curl};\Omega)$ such that

$$\hat{a}(\boldsymbol{E},\boldsymbol{\psi}) = (\boldsymbol{F},\boldsymbol{\psi}) \qquad \forall\ \boldsymbol{\psi} \in H_0(\mathbf{curl};\Omega), \tag{2.3}$$

where $(\cdot,\cdot)$ denotes the inner product in $(L^2(\Omega))^3$ and

$$\hat{a}(\boldsymbol{E},\boldsymbol{\psi}) = (\nabla\times\boldsymbol{E},\nabla\times\boldsymbol{\psi}) - \omega^2((\epsilon + i\sigma/\omega)\boldsymbol{E},\boldsymbol{\psi}). \tag{2.4}$$

In order to ensure the well-posedness of the variational problem (2.3), we shall make the following assumptions throughout this paper:

$$\sigma > 0 \ \text{ in } \overline{\Omega}, \tag{2.5}$$

or

$$\sigma = 0, \text{ and } \omega^2\epsilon \text{ is not an eigenvalue of equations (1.1)-(1.2)}. \tag{2.6}$$

Assume that Ω is discretized into a regular tetrahedron mesh $\mathcal{T}_h$, where h is the maximum diameter of the tetrahedron in $\mathcal{T}_h$. We introduce the following Nédélec's first type elements space $\boldsymbol{V}^{k,1}(\mathcal{T}_h)$ and second type elements space $\boldsymbol{V}^{k,2}(\mathcal{T}_h)$

$$\begin{aligned}
\boldsymbol{V}^{k,1}(\mathcal{T}_h) &= \Big\{\boldsymbol{v}_h \in H_0(\mathbf{curl};\Omega) \ \Big|\ \boldsymbol{v}_h|_\tau \in (\mathcal{P}_{k-1})^3 \oplus \Big\{\boldsymbol{p} \in (\tilde{\mathcal{P}}_k)^3 \mid \boldsymbol{x}\cdot\boldsymbol{p} = 0\Big\} \ \text{ for all } \tau \in \mathcal{T}_h\Big\},\\
\boldsymbol{V}^{k,2}(\mathcal{T}_h) &= \Big\{\boldsymbol{v}_h \in H_0(\mathbf{curl};\Omega) \ \Big|\ \boldsymbol{v}_h|_\tau \in (\mathcal{P}_k)^3 \ \text{ for all } \tau \in \mathcal{T}_h\Big\},
\end{aligned}$$

where $\mathcal{P}_k$ is the space of polynomials of total degree at most k and $\tilde{\mathcal{P}}_k$ the space of homogeneous polynomials of order k.

Using the degrees of freedom of Nédélec edge element space $\boldsymbol{V}^{k,l}(\mathcal{T}_h)(l = 1,2)$, we can define the corresponding edge interpolations $\Pi_h^{\mathbf{curl},l}\boldsymbol{u} \in \boldsymbol{V}^{k,l}(\mathcal{T}_h)$, for any $\boldsymbol{u} \in \boldsymbol{H}^{1/2+\bar{\delta}}(\mathbf{curl};\tau)$ with constant $\bar{\delta} > 0$ and $\tau \in \mathcal{T}_h$ (c.f. [9, 10]), where the Sobolev space $\boldsymbol{H}^s(\mathbf{curl};\tau) = \{\boldsymbol{u} \in (H^s(\tau))^3, \nabla\times\boldsymbol{u} \in (H^s(\tau))^3\}$ $(s > 0)$ associated with the norm $\|\boldsymbol{u}\|_{\boldsymbol{H}^s(\mathbf{curl};\tau)} = (\|\boldsymbol{u}\|^2_{(H^s(\tau))^3} + \|\nabla\times\boldsymbol{u}\|^2_{(H^s(\tau))^3})^{1/2}$.

The next lemma states the interpolation error estimate, see Theorem 8.15 in [8], Lemma 3.2 and Lemma 3.3 in [2].

Lemma 2.1. *Let $\boldsymbol{V}^{k,2}(\mathcal{T}_h)$ and $\Pi_h^{\mathbf{curl},2}$ be constructed as above. Then*

(1) *If $\boldsymbol{u} \in (H^{s+1}(\Omega))^3$ for $1 \le s \le k$, we have*

$$\|\boldsymbol{u} - \Pi_h^{\mathbf{curl},2}\boldsymbol{u}\|_0 + h\|\nabla \times (\boldsymbol{u} - \Pi_h^{\mathbf{curl},2}\boldsymbol{u})\|_0 \lesssim h^{s+1}\|\boldsymbol{u}\|_{(H^{s+1}(\Omega))^3}. \tag{2.7}$$

(2) *If $\boldsymbol{u} \in \boldsymbol{H}^\delta(\mathbf{curl};\Omega)$ for $1/2 < \delta \le 1$, we have*

$$\|\boldsymbol{u} - \Pi_h^{\mathbf{curl},2}\boldsymbol{u}\|_{H(\mathbf{curl};\Omega)} \lesssim h^\delta\|\boldsymbol{u}\|_{\boldsymbol{H}^\delta(\mathbf{curl};\Omega)}, \tag{2.8}$$

The estimate of (2.8) also holds for the interpolation $\Pi_h^{\mathbf{curl},1}$.

To save notation, we shall use $\boldsymbol{V}(\mathcal{T}_h)$ for both first and second types Nédélec element spaces. The edge finite element approximation of (2.3) is: Find $\boldsymbol{E}_h \in \boldsymbol{V}(\mathcal{T}_h)$ such that

$$\hat{a}(\boldsymbol{E}_h, \boldsymbol{\psi}_h) = (\boldsymbol{F}, \boldsymbol{\psi}_h) \quad \forall\ \boldsymbol{\psi}_h \in \boldsymbol{V}(\mathcal{T}_h). \tag{2.9}$$

In the rest of this paper, we shall assume the mesh size h is small enough such that there exists a unique solution to (2.9).

2.1. **Two-grid Method.** We start with two regular tetrahedral meshes: $\mathcal{T}_h$ and $\mathcal{T}_H$, with different mesh sizes h and H such that $h \ll H$. The two-grid method is to solve the original indefinite problem in a coarse mesh space $\boldsymbol{V}^{k,2}(\mathcal{T}_H)$ and then correct the approximation by solving a SPD problem in the fine space $\boldsymbol{V}^{k,2}(\mathcal{T}_h)$ for which we can use standard multigrid method or preconditions. Note that $h \ll H$, the indefinite problem to be solved is in a small size and the computation cost is negligible.

Algorithm 2.2 (Two-Grid Method)**.**

(1) *Find $\boldsymbol{E}_H \in \boldsymbol{V}^{k,2}(\mathcal{T}_H)$ such that*

$$\hat{a}(\boldsymbol{E}_H, \boldsymbol{\psi}_H) = (\boldsymbol{F}, \boldsymbol{\psi}_H) \ \ \forall\ \boldsymbol{\psi}_H \in \boldsymbol{V}^{k,2}(\mathcal{T}_H). \tag{2.10}$$

(2) *Find $\boldsymbol{E}^h \in \boldsymbol{V}^{k,2}(\mathcal{T}_h)$ such that*

$$a(\boldsymbol{E}^h, \boldsymbol{\psi}_h) = (\boldsymbol{F}, \boldsymbol{\psi}_h) + N(\boldsymbol{E}_H, \boldsymbol{\psi}_h) \quad \forall\ \boldsymbol{\psi}_h \in \boldsymbol{V}^{k,2}(\mathcal{T}_h), \tag{2.11}$$

where

$$a(\boldsymbol{E}, \boldsymbol{\psi}) = (\nabla \times \boldsymbol{E}, \nabla \times \boldsymbol{\psi}) + (\boldsymbol{E}, \boldsymbol{\psi}), \quad \textit{and} \quad N(\boldsymbol{E}, \boldsymbol{\psi}) = \omega^2((\epsilon + i\sigma/\omega)\boldsymbol{E}, \boldsymbol{\psi}) - (\boldsymbol{E}, \boldsymbol{\psi}).$$

The following theorem ensures that the approximation $\boldsymbol{E}^h$ obtained by the two-grid methods is sufficiently close to the finite element solution $\boldsymbol{E}_h$ of (2.9).

Theorem 2.3. *Under the hypotheses of Lemma 2.4, and furthermore assume the solution of (2.3) $\boldsymbol{E} \in (H^{s+1}(\Omega))^3$ for some $s \in [1, k]$. Let $\boldsymbol{E}^h$ and $\boldsymbol{E}_h \in \boldsymbol{V}^{k,2}(\mathcal{T}_h)$ be the approximation obtained by Algorithm 2.2 and (2.9), respectively. Then there exist a constant $\delta \in (1/2, 1]$ and a constant $h_0 > 0$, such that for all $h < h_0$, we have*

$$\|\boldsymbol{E}_h - \boldsymbol{E}^h\|_{H(\mathbf{curl};\Omega)} \lesssim H^{s+\delta}|\boldsymbol{E}|_{(H^{s+1}(\Omega))^3}, \tag{2.12}$$

$$\|\boldsymbol{E} - \boldsymbol{E}^h\|_{H(\mathbf{curl};\Omega)} \lesssim (h^s + H^{s+\delta})|\boldsymbol{E}|_{(H^{s+1}(\Omega))^3}. \tag{2.13}$$

where the constants in (2.12) and (2.13) depend only on Ω, the shape-regularity of the meshes, the parameters ϵ and σ.

Note to mention that L^2 error estimates is instrumental in developing and analyzing the two-grid method. In the following we only present the corresponding results and list an outline of its proof in the next subsection.

Lemma 2.4. *Suppose (2.3) and (2.9) are well posed and* $\boldsymbol{E}, \boldsymbol{E}_h$ *are solutions to (2.3) and (2.9), respectively. Then there exists a constant* $h_0 > 0$*, such that for all* $h < h_0$*,*

$$\|\boldsymbol{E} - \boldsymbol{E}_h\|_{H(\mathbf{curl};\Omega)} \lesssim \inf_{\boldsymbol{v}_h \in \boldsymbol{V}(\mathcal{T}_h)} \|\boldsymbol{E} - \boldsymbol{v}_h\|_{H(\mathbf{curl};\Omega)}. \tag{2.14}$$

Furthermore, there exist a constant $\delta \in (1/2, 1]$ *depending only the shape of* Ω *(with* $\delta = 1$ *when the domain* Ω *is convex), such that for all* $h < h_0$*,*

$$\|\boldsymbol{E} - \boldsymbol{E}_h\|_0 \lesssim \inf_{\boldsymbol{v}_h \in \boldsymbol{V}(\mathcal{T}_h)} \left(\|\boldsymbol{E} - \boldsymbol{v}_h\|_0 + h^{\delta}\|\nabla \times (\boldsymbol{E} - \boldsymbol{v}_h)\|_0\right). \tag{2.15}$$

where the above constants in (2.14) and (2.15) depend only on Ω*, the shape-regularity of the meshes, the parameters* ϵ *and* σ*.*

As a consequence of Lemma 2.4 and (2.7), we obtain *a priori* optimal order error estimate in L^2-norm.

Corollary 2.5. *Suppose the assumptions in Lemma 2.4 hold and furthermore assume* $\boldsymbol{E} \in (H^{s+1}(\Omega))^3$ *for some* $s \in [1, k]$ *and* $\boldsymbol{E}_h \in \boldsymbol{V}^{k,2}(\mathcal{T}_h)$*. Then there exists a constant* C_3 *depending only on* Ω*, the shape-regularity of the meshes, the parameters* ϵ *and* σ*, such that for all* $h < h_0$*, such that for all* $h < h_0$*, we have*

$$\|\boldsymbol{E} - \boldsymbol{E}_h\|_0 \lesssim h^{s+\delta}\|\boldsymbol{E}\|_{(H^{s+1}(\Omega))^3},$$

where h_0 *and* δ *are given in Lemma 2.4.*

Proof of Theorem 2.3. By (2.9) (2.11), the definitions of $a(\cdot,\cdot)$ and $N(\cdot,\cdot)$, we have

$$\begin{aligned} a(\boldsymbol{E}_h - \boldsymbol{E}^h, \boldsymbol{\psi}_h) &= -N(\boldsymbol{E}_h - \boldsymbol{E}_H, \boldsymbol{\psi}_h) \\ &\lesssim \|\boldsymbol{E}_h - \boldsymbol{E}_H\|_0 \|\boldsymbol{\psi}_h\|_0 \\ &\lesssim (\|\boldsymbol{E} - \boldsymbol{E}_h\|_0 + \|\boldsymbol{E} - \boldsymbol{E}_H\|_0) \|\boldsymbol{\psi}_h\|_{H(\mathbf{curl};\Omega)} \end{aligned}$$

Picking $\boldsymbol{\psi}_h = \boldsymbol{E}_h - \boldsymbol{E}^h$ in the above inequality and noting that $a(\cdot,\cdot) = \|\cdot\|^2_{H(\mathbf{curl};\Omega)}$, we obtain

$$\|\boldsymbol{E}_h - \boldsymbol{E}^h\|_{H(\mathbf{curl};\Omega)} \lesssim \|\boldsymbol{E} - \boldsymbol{E}_h\|_0 + \|\boldsymbol{E} - \boldsymbol{E}_H\|_0. \tag{2.16}$$

Using Corollary 2.5 and the assumption $h \ll H$ in (2.16), we get

$$\begin{aligned} \|\boldsymbol{E}_h - \boldsymbol{E}^h\|_{H(\mathbf{curl};\Omega)} &\lesssim (h^{s+1} + H^{s+1})|\boldsymbol{E}|_{(H^{s+1}(\Omega))^3} \\ &\lesssim H^{s+1}|\boldsymbol{E}|_{(H^{k+1}(\Omega))^3}, \end{aligned}$$

Using the triangular inequality and the above inequality, we obtain

$$\begin{aligned} \|\boldsymbol{E} - \boldsymbol{E}^h\|_{H(\mathbf{curl};\Omega)} &\lesssim \|\boldsymbol{E} - \boldsymbol{E}_h\|_{H(\mathbf{curl};\Omega)} + \|\boldsymbol{E}_h - \boldsymbol{E}^h\|_{H(\mathbf{curl};\Omega)} \\ &\lesssim (h^s + H^{s+1})|\boldsymbol{E}|_{(H^{s+1}(\Omega))^3} \end{aligned}$$

which concludes the proof. □

From Theorem 2.3, to obtain an optimal order approximation from the two-grid method, we can choose $H = O(h^{\lambda})$, such that $H^{s+\delta} = h^s$. When the domain is convex, we can choose $H = h^{1/2}$. As an example, when $H = 1/8$, $\dim \boldsymbol{V}^{1,2}(\mathcal{T}_h) = 7{,}322{,}562$ while $\dim \boldsymbol{V}^{1,2}(\mathcal{T}_H) = 8{,}802$ is much smaller. Note that the main computation cost is to solve a SPD problem in the fine space, and we can use the new preconditioner of Hiptmair and Xu [4].

While the two-grid method can be used as a discretization method applied directly to (1.1)-(1.2), it can also be applied to design efficient iterative methods for solving (1.1)-(1.2) that is directly discretized by a finite element method. Two approaches are possible. One is to use it to devise a multigrid method in which smoothers are only carried using symmetric positive definite system like (2.11) while only the original discretizated system (2.10) is solved on the coarsest grid. Another approach is to extend a result of Hiptmair and Xu [4] to design a preconditioner (that involves four Poisson solvers a coarse grid solver on the original system, and some local symmmetric positive definite smoothers on the finest grids).

Now we present numerical experiments of Algorithm 2.2 to support our theoretical results. We choose $\Omega = (0,1)^3$ and discretize it by a hierarchy of multilevel uniform cubic meshes. Each tetrahedron mesh is obtained by dividing every cube into six sub-tetrahedra. We use the lowest order second family of Nédélec edge elements. The right-hand side of (2.9) is chosen so that the true solution is: $U(x,y,z) = [y(1-y)z(1-z), x(1-x)z(1-z), x(1-x)y(1-y)]$.

H	h	e	$e * H^{-2}$
1/2	1/4	8.66e-2	0.346
1/3	1/9	3.90e-2	0.351
1/4	1/16	2.20e-2	0.352
1/5	1/25	1.41e-2	0.352

H	h	e	$e * H^{-2}$
1/5	1/25	1.19e-1	2.975
1/6	1/36	7.49e-2	2.696
1/7	1/49	5.20e-2	2.548
1/8	1/64	3.85e-2	2.464

H	h	e	$e * H^{-2}$
1/6	1/36	1.38e-1	4.968
1/7	1/49	8.92e-2	4.371
1/8	1/64	6.93e-2	4.435

TABLE 1. From left to right: error of two-grid approximation for $\omega = 1$, $\omega = 5$, $\omega = 10$.

In the above table, e denotes $\|\boldsymbol{E} - \boldsymbol{E}^h\|_{H(\mathbf{curl};\Omega)}$. Since $e * H^2$ approaches to a constant, we conclude $\|\boldsymbol{E} - \boldsymbol{E}^h\|_{H(\mathbf{curl};\Omega)}$ is second order. Note that for larger ω, we need more refined coarse grid which reflects to the assumption: the mesh size is small enough.

2.2. **Optimal L^2 error estimates.** One key to the proof of Lemma 2.4 is to transform the L^2 error estimates into the L^2 estimate of a discrete divergence-free function which belongs to the edge finite element spaces, and then use the approximation of the discrete divergence-free function by the continuous divergence-free function and a duality argument for the continuous divergence-free function. In the following, we only list an outline of our proof.

For any $\boldsymbol{v}_h \in \boldsymbol{V}(\mathcal{T}_h)$, using the discrete Helmholtz decompositions for $\boldsymbol{v}_h - \boldsymbol{E}_h$, we have

$$\boldsymbol{v}_h - \boldsymbol{E}_h = \boldsymbol{w}_h + \nabla q_h, \tag{2.17}$$

where $\boldsymbol{w}_h \in \boldsymbol{V}_0(\mathcal{T}_h) := \{\boldsymbol{u}_h \in \boldsymbol{V}(\mathcal{T}_h) \mid (\boldsymbol{u}_h, \nabla p_h) = 0 \text{ for } \forall\ p_h \in S_h\}$ and $\boldsymbol{q}_h \in S_h := \{\ p_h \in H_0^1(\Omega) \cap C(\bar{\Omega}) \big|\ p_h|_\tau \in \mathcal{P}_{k+l-1},\ \text{for } \forall\ \tau \in \mathcal{T}_h, l = 1,2\}$.

Then using the Galerkin orthogonality and noting that that $\boldsymbol{E} - \boldsymbol{E}_h$ is discrete divergence-free, we conclude the following Lemma.

Lemma 2.6. *The following estimate holds*

$$\|\boldsymbol{E} - \boldsymbol{E}_h\|_0 \leq \sqrt{2}\left(\|\boldsymbol{E} - \boldsymbol{v}_h\|_0 + \|\boldsymbol{w}_h\|_0\right). \tag{2.18}$$

Combining Lemma 2.6 and the Galerkin orthogonality again, we have

Lemma 2.7. *The following estimate holds*

$$\|\nabla \times \boldsymbol{e}_h\|_0 \lesssim \|\boldsymbol{E} - \boldsymbol{v}_h\|_{H(\mathbf{curl};\Omega)} + \|\boldsymbol{w}_h\|_0, \tag{2.19}$$

where the constant only depends on the parameters α *and* β.

It is clear that it suffices to estimate $\|\boldsymbol{w}_h\|_0$ for completing error estimates in both L^2-norm and $\boldsymbol{H}(\mathbf{curl})$-seminorm. Noting that $\boldsymbol{w}_h$ is discrete divergence-free, and the following lemma shows that the discrete divergence-free function can be well approximated by a continuous divergence-free function.

Lemma 2.8 (see [3]). *For any given* $\boldsymbol{u}_h \in \boldsymbol{V}_0(\mathcal{T}_h)$, *there exists a* $\boldsymbol{u} \in H_0(\mathbf{curl};\Omega)$ *that satisfies*

$$\nabla \times \boldsymbol{u} = \nabla \times \boldsymbol{u}_h, \ \nabla \cdot \boldsymbol{u} = 0, \tag{2.20}$$

and

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_0 \lesssim h^\delta \|\nabla \times \boldsymbol{u}_h\|_0 \tag{2.21}$$

with a constant $\delta \in (1/2, 1]$ *and* $\delta = 1$ *for the case that* Ω *is smooth or convex.*

In the following, we use Lemma 2.8 and a duality argument for the continuous divergence-free function for obtaining $\|\boldsymbol{w}_h\|_0$.

Lemma 2.9. *There exists a constant* $h_0 > 0$ *independents of* h, $\boldsymbol{E}$ *and* $\boldsymbol{E}_h$, *such that for all* $h < h_0$, *we have*

$$\|\boldsymbol{w}_h\|_0 \lesssim \|\boldsymbol{E} - \boldsymbol{v}_h\|_0 + h^\delta \|\nabla \times (\boldsymbol{E} - \boldsymbol{v}_h)\|_0, \tag{2.22}$$

where the constant δ *is the exponent in Lemma 2.8.*

Proof. For given $\boldsymbol{w}_h \in \boldsymbol{V}_0(\mathcal{T}_h)$ in (2.17), using Lemma 2.8, there exists a $\boldsymbol{w} \in H_0(\mathbf{curl};\Omega)$ satisfies

$$\nabla \times \boldsymbol{w} = \nabla \times \boldsymbol{w}_h, \quad \nabla \cdot \boldsymbol{w} = 0, \tag{2.23}$$

and

$$\|\boldsymbol{w} - \boldsymbol{w}_h\|_0 \lesssim h^\delta \|\nabla \times \boldsymbol{w}_h\|_0. \tag{2.24}$$

Taking the **curl** of both side of (2.17), we have

$$\nabla \times \boldsymbol{w}_h = \nabla \times (\boldsymbol{v}_h - \boldsymbol{E}_h). \tag{2.25}$$

Using (2.24), (2.25) and the triangle inequality, we have

$$\begin{aligned} \|\boldsymbol{w} - \boldsymbol{w}_h\|_0 &\lesssim h^\delta \|\nabla \times (\boldsymbol{v}_h - \boldsymbol{E}_h)\|_0 \\ &\le h^\delta \left(\|\nabla \times \boldsymbol{e}_h\|_0 + \|\nabla \times (\boldsymbol{E} - \boldsymbol{v}_h)\|_0 \right). \end{aligned} \tag{2.26}$$

Using the triangle inequality, (2.26) and (2.19), we obtain

$$\begin{aligned} \|\boldsymbol{w}_h\|_0 &\le C_1 \left(h^\delta \left(\|\nabla \times \boldsymbol{e}_h\|_0 + \|\nabla \times (\boldsymbol{E} - \boldsymbol{v}_h)\|_0 \right) + \|\boldsymbol{w}\|_0 \right) \\ &\le C_2 \left(h^\delta \|\boldsymbol{E} - \boldsymbol{v}_h\|_{H(\mathbf{curl};\Omega)} + h^\delta \|\boldsymbol{w}_h\|_0 + \|\boldsymbol{w}\|_0 \right). \end{aligned}$$

where the constants $C_i (i = 1, 2)$ independent of h, $\boldsymbol{E}$ and $\boldsymbol{E}_h$. Choosing $h_1 > 0$ satisfies $1 - C_2 h_1^\delta > 0$, then for all $h < h_1$, we have

$$\|\boldsymbol{w}_h\|_0 \lesssim h^\delta \|\boldsymbol{E} - \boldsymbol{v}_h\|_{H(\mathbf{curl};\Omega)} + \|\boldsymbol{w}\|_0. \tag{2.27}$$

Next, we will use a duality argument to obtain the L^2 estimate of $\boldsymbol{w}$.

Let $\boldsymbol{\phi} \in H_0(\mathbf{curl};\Omega)$ solves the following auxiliary problem

$$\hat{a}(\boldsymbol{\psi}, \boldsymbol{\phi}) = (\boldsymbol{w}, \boldsymbol{\psi}) \quad \forall\, \boldsymbol{\psi} \in H_0(\mathbf{curl};\Omega). \tag{2.28}$$

Taking $\boldsymbol{\psi} = \nabla q$ with some $q \in H_0^1(\Omega)$ in (2.28), and using the Green formula with the fact $\nabla \cdot \boldsymbol{w} = 0$, we have

$$((\alpha + i\beta)\nabla q, \boldsymbol{\phi}) = 0. \tag{2.29}$$

Since $\nabla \cdot \boldsymbol{w} = 0$, then we have the following regularity result for auxiliary problem (2.28) (See [7])

$$\|\boldsymbol{\phi}\|_{H^\delta(\mathbf{curl};\Omega)} \lesssim \|\boldsymbol{w}\|_0. \tag{2.30}$$

Combining (2.25) and (2.23), we have

$$\nabla \times (\boldsymbol{w} - (\boldsymbol{v}_h - \boldsymbol{E}_h)) = 0. \tag{2.31}$$

Noting that $\boldsymbol{w} - (\boldsymbol{v}_h - \boldsymbol{E}_h) \in H_0(\mathbf{curl};\Omega)$ and (2.31), thus using the exact sequence property, there exists a $p \in H_0^1(\Omega)$, such that

$$\boldsymbol{w} - (\boldsymbol{v}_h - \boldsymbol{E}_h) = \nabla p. \tag{2.32}$$

Using (2.31), (2.32) and (2.29), we have

$$\begin{aligned} \hat{a}(\boldsymbol{w} - (\boldsymbol{v}_h - \boldsymbol{E}_h), \boldsymbol{\phi}) &= (\nabla \times (\boldsymbol{w} - (\boldsymbol{v}_h - \boldsymbol{E}_h)), \nabla \times \boldsymbol{\phi}) \\ &\quad - ((\alpha + i\beta)(\boldsymbol{w} - (\boldsymbol{v}_h - \boldsymbol{E}_h)), \boldsymbol{\phi}) \\ &= ((\alpha + i\beta)\nabla p, \boldsymbol{\phi}) = 0 \end{aligned} \tag{2.33}$$

Setting $\boldsymbol{\psi} = \boldsymbol{w}$ in (2.28), and from (2.33), (2.29), Galerkin orthogonality, (2.28), (2.8) and (2.30), we have

$$\begin{aligned} \|\boldsymbol{w}\|_0^2 &= \hat{a}(\boldsymbol{w}, \boldsymbol{\phi}) = \hat{a}(\boldsymbol{w} - (\boldsymbol{v}_h - \boldsymbol{E}_h), \boldsymbol{\phi}) + \hat{a}(\boldsymbol{v}_h - \boldsymbol{E}_h, \boldsymbol{\phi}) \\ &= \hat{a}(\boldsymbol{e}_h, \boldsymbol{\phi}) - \hat{a}(\boldsymbol{E} - \boldsymbol{v}_h, \boldsymbol{\phi}) \\ &= \hat{a}(\boldsymbol{e}_h, \boldsymbol{\phi} - \boldsymbol{\Pi}_h^{curl}\boldsymbol{\phi}) - (\boldsymbol{w}, \boldsymbol{E} - \boldsymbol{v}_h) \\ &\lesssim \|\boldsymbol{e}_h\|_{H(\mathbf{curl};\Omega)}\|\boldsymbol{\phi} - \boldsymbol{\Pi}_h^{\mathbf{curl}}\boldsymbol{\phi}\|_{H(\mathbf{curl};\Omega)} + \|\boldsymbol{w}\|_0\|\boldsymbol{E} - \boldsymbol{v}_h\|_0 \\ &\lesssim \|\boldsymbol{w}\|_0 \left(h^\delta\|\boldsymbol{e}_h\|_{H(\mathbf{curl};\Omega)} + \|\boldsymbol{E} - \boldsymbol{v}_h\|_0\right), \end{aligned}$$

where $\boldsymbol{\Pi}_h^{\mathbf{curl}}$ denotes $\Pi_h^{\mathbf{curl},1}$ (or $\Pi_h^{\mathbf{curl},2}$). Hence we concludes that

$$\|\boldsymbol{w}\|_0 \lesssim \|\boldsymbol{E} - \boldsymbol{v}_h\|_0 + h^\delta\|\boldsymbol{e}_h\|_{H(\mathbf{curl};\Omega)}. \tag{2.34}$$

At last, substituting (2.34) into (2.27) and using Lemmas 2.6 and 2.7, we obtain

$$\begin{aligned} \|\boldsymbol{w}_h\|_0 &\le C_3\left(\|\boldsymbol{E} - \boldsymbol{v}_h\|_0 + h^\delta\|\nabla \times (\boldsymbol{E} - \boldsymbol{v}_h)\|_0 + h^\delta\|\boldsymbol{e}_h\|_{H(\mathbf{curl};\Omega)}\right) \\ &\le C_4\left(\|\boldsymbol{E} - \boldsymbol{v}_h\|_0 + h^\delta\|\nabla \times (\boldsymbol{E} - \boldsymbol{v}_h)\|_0 + h^\delta\|\boldsymbol{w}_h\|_0\right). \end{aligned}$$

where the constants $C_i (i = 3, 4)$ independent of h, $\boldsymbol{E}$ and $\boldsymbol{E}_h$. Choosing $h_2 > 0$ satisfies $1 - C_4 h_2^\delta > 0$, then for all $h < h_0 := \min\{h_1, h_2\}$, we obtain the estimate (2.22). □

Lemma 2.1 is a direct consequence of Lemmas 2.6, 2.7 and 2.9.

2.3. **Adaptive edge finite element method.** In view of theoretical analysis, the finite element solution of indefinite time-harmonic Maxwell's equations exists uniquely and the corresponding error estimates have been proved only provided that the mesh size is sufficiently small. Furthermore, the mesh size strongly depend on the angular frequency ω. Noting that mesh adaptation can present the most efficient distributions of degrees of freedom and yield sufficiently accurate solutions using the fewest free parameters. So in this subsection, we will present some results on adaptive finite element methods for (1.1)-(1.2) with $\sigma = 0$. Here, we only consider the case that $\boldsymbol{V}(\mathcal{T}_h) = \boldsymbol{V}^{1,l}(\mathcal{T}_h)(l = 1, 2)$.

We use standard AEFEM procedure which is analogous to AFEM procedure in Cascon *et. al.* [1] with the following *a posteriori* error estimator:

$$\eta^2(\boldsymbol{E}_k, \boldsymbol{F}, \mathcal{T}_k) = \sum_{\tau\in\mathcal{T}_k} h_\tau^2 \left(\|R_1(\boldsymbol{E}_k)\|_{0,\tau}^2 + \|R_2(\boldsymbol{E}_k)\|_{0,\tau}^2\right) + \sum_{f\in\mathcal{F}(\mathcal{T}_k)} h_f \left(\|J_1(\boldsymbol{E}_k)\|_{0,f}^2 + \|J_2(\boldsymbol{E}_k)\|_{0,f}^2\right),$$

where $h_\tau = |\tau|^{1/3}$ measures the local mesh size of the element τ, $h_f := 1/2(h_{\tau_1} + h_{\tau_2})$ for any $f \in \mathcal{F}(\mathcal{T}_k)$ sharing by two elements τ_1 and τ_2, $R_1(\boldsymbol{E}_k) := \boldsymbol{F} - \nabla\times(\nabla\times\boldsymbol{E}_k) + \omega^2\boldsymbol{E}_k$, $R_2(\boldsymbol{E}_k) := \boldsymbol{F} + \omega^2\boldsymbol{E}_k$, $J_1(\boldsymbol{E}_k) := [\![(\nabla\times\boldsymbol{E}_k)\times\boldsymbol{n}_f]\!]$, $J_2(\boldsymbol{E}_k) := [\![(\boldsymbol{F}+\omega^2\boldsymbol{E}_k)\cdot\boldsymbol{n}_f]\!]$, $[\![\boldsymbol{u}]\!]$ is the interelement jumps across f of a function $\boldsymbol{u}$ and $\mathcal{F}(\mathcal{T}_k)$ is the set of interior faces of $\mathcal{T}_k$.

We define the oscillation as follows

$$\begin{aligned}\mathrm{osc}^2(\boldsymbol{E}_k, \boldsymbol{F}, \mathcal{T}_k) \;=\;& h_\tau^2\left(\|(Id - Q_{\mathcal{T}_k})R_1(\boldsymbol{E}_k)\|_{0,\tau}^2 + \|(Id - Q_{\mathcal{T}_k})R_2(\boldsymbol{E}_k)\|_{0,\tau}^2\right)\\ &+ \sum_{f\in\tau\cap\mathcal{F}(\mathcal{T}_k)} h_f\left(\|(Id - Q_{\mathcal{T}_k})J_1(\boldsymbol{E}_k)\|_{0,f}^2 + \|(Id - Q_{\mathcal{T}_k})J_2(\boldsymbol{E}_k)\|_{0,f}^2\right),\end{aligned}$$

where the operator $Q_{\mathcal{T}_k}$ is the L^2 projection onto the set of piecewise $(\mathcal{P}_0)^3$ or $\mathcal{P}_0$ over $\tau \in \mathcal{T}$ or $f \in \mathcal{F}(\mathcal{T}_k)$.

Let $\{\mathcal{T}_k, \boldsymbol{V}(\mathcal{T}_k), \boldsymbol{E}_k, \eta_k, \mathrm{osc}_k\}_{k\geq 0}$ be the sequence of meshes, finite element spaces, and discrete solutions, estimators and oscillations produced by AEFEM in the k-th step.

Theorem 2.10. *There exists constants $C > 0$, $\delta \in (0, 1)$, and $\tilde{h}_0$ depending only on Ω and the parameter ω, such that for all $h_k < \tilde{h}_0$, we have,*

$$\|\boldsymbol{E} - \boldsymbol{E}_{k+1}\|_{H(\mathbf{curl};\Omega)}^2 + C\eta_{k+1}^2 \leq \delta\left(\|\boldsymbol{E} - \boldsymbol{E}_k\|_{H(\mathbf{curl};\Omega)}^2 + C\eta_k^2\right). \tag{2.35}$$

Suppose $(\boldsymbol{E}, \boldsymbol{F}) \in \mathcal{A}_s$, where $\mathcal{A}_s$ denote some approximation space. Then there exist a constant $C(\mathcal{T}_0, \omega)$ depending only on the shape of triangles in $\mathcal{T}_0$ and parameter ω, such that

$$\left(\|\boldsymbol{E} - \boldsymbol{E}_k\|_{H(\mathbf{curl};\Omega)}^2 + C\mathrm{osc}_k^2\right)^{1/2} \leq C(\mathcal{T}_0, \omega)\left(\#\mathcal{T}_k - \#\mathcal{T}_0\right)^{-s}. \tag{2.36}$$

The contraction (2.35) between two consecutive steps, hence AEFEM will converge in finite steps for a given tolerance. Furthermore according to (2.36) it produce the best possible approximation.

The main difficulties of the proof of Theorem 2.10 are a quasi-orthogonality and a localized upper bound of *a posteriori* error estimate. Due to space limitation, we only present the corresponding Lemmas and key techniques.

Lemma 2.11 (quasi-orthogonality)**.** *for any $\delta_0 \in (0, 1)$, there exists a constant $h(\delta_0)$ solely depending on the parameter ω and the domain Ω, such that, if $h_{\mathcal{T}} \leq h(\delta_0)$, we have*

$$\|\boldsymbol{E} - \boldsymbol{E}_{k+1}\|_{\mathbf{curl},\Omega}^2 \leq \Lambda_0\|\boldsymbol{E} - \boldsymbol{E}_k\|_{\mathbf{curl},\Omega}^2 - \|\boldsymbol{E}_{k+1} - \boldsymbol{E}_k\|_{\mathbf{curl},\Omega}^2, \tag{2.37}$$

where the constant $\Lambda_0 := \frac{1}{1-\delta_0}$.

We use some techniques which developed in the aforementioned L^2 error estimates for proving Lemma 2.11.

Lemma 2.12 (Localized upper bound)**.** *Let $\mathcal{R}_{\mathcal{T}_k \to \mathcal{T}_{k+1}}$ be the set of refined elements from $\mathcal{T}_k$ to $\mathcal{T}_{k+1}$, then we have*

$$\|\boldsymbol{E}_{k+1} - \boldsymbol{E}_k\|_{\mathbf{curl},\Omega} \lesssim \eta(\boldsymbol{E}_k, \boldsymbol{F}, \mathcal{R}_{\mathcal{T}_k \to \mathcal{T}_{k+1}}), \tag{2.38}$$

where the constant depends only on the ratio of two levels grids.

The above lemma shows that the error can be estimated by using only the indicators of refined elements without a buffer layer. The main technique is to use some appropriate interpolation operators. Due to a technical reason, we assume that, at most finite steps of refinements separate two consecutive subdivisions.

At the end of subsection, we present a numerical example to support our theoretical results. We choose a "L-shaped" domain $\Omega = [-1,1]^3/(0,1] \times (0,1] \times [-1,-1]$, the finite element space $\boldsymbol{V}(\mathcal{T}_h) = \boldsymbol{V}^{1,1}(\mathcal{T}_h)$, and the true solution is $\boldsymbol{E} = \mathbf{grad}(r^{\frac{2}{3}}\sin(\frac{2}{3}\theta))$ in cylindrical coordinates. Figure 1 shows the robustness and quasi-optimality of adaptive mesh refinements of the error $\|\boldsymbol{E} - \boldsymbol{E}_k\|_{H(\mathbf{curl};\Omega)}$ for $\theta \in (0.1, 0.5)$ in different values of ω.

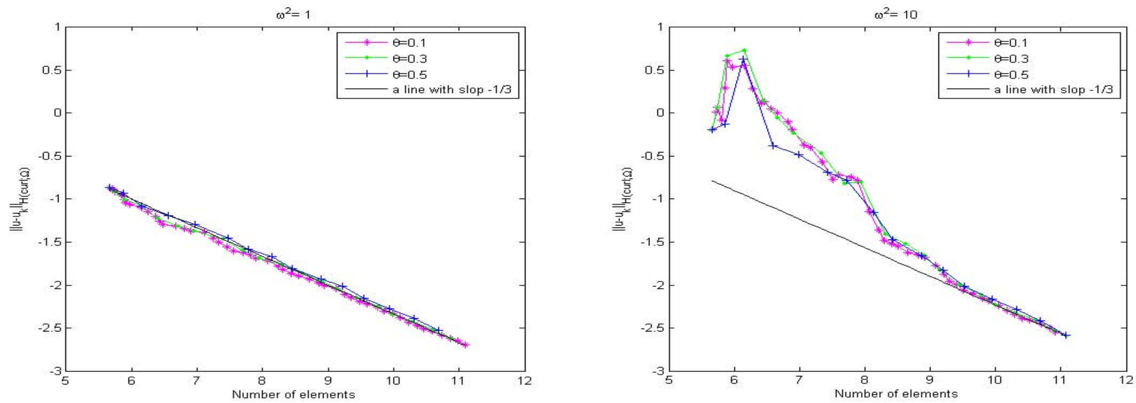

FIGURE 1. Quasi-optimality of adaptive mesh refinements of the error $\|\boldsymbol{E} - \boldsymbol{E}_k\|_{H(\mathbf{curl};\Omega)}$ for different values of ω and θ.

Acknowledgments. The author would like to thank my Ph.D. advisors Prof. J. Xu and Prof. S. Shu, collaborators Prof. G. Wittum and Prof. L. Chen for their constructive comments, and Prof. R. Hiptmair for helpful discussions on optimal L^2 error estimates.

References

[1] J. Cascon, C. Kreuzer, R. Nochetto and K. Siebert, Quasi-optimal convergence rate for an adaptive finite element method. *SIAM J. Numer. Anal.*, **46** (2008), 2524-2550.

[2] P. Ciarlet and J. Zou, Fully discrete finite element approaches for time-dependent Maxwell's equations, *Numer. Math.*, **82** (1999), 193-219.

[3] R. Hiptmair, Finite elements in computational electromagnetism. *Acta. Numer.*, **11** (2002), 237-239.

[4] R. Hiptmair and J. Xu, Nodal auxiliary spaces preconditions in H(curl) and H(div) spaces. *SIAM J. Numer. Anal.*, **45** (2007), 2483 - 2509

[5] R. Hoppe and J. Schoberl. Convergence of adaptive edge element methods for the 3d eddy currents equations. (2009), to appear in *J. Comp. Math.*
[6] P. Monk, A finite element methods for approximating the time-harmonic Maxwell equations. *Numer. Math.*, **63** (1992), 243-261.
[7] P. Monk, A simple proof of convergence for an edge element discretization of Maxwell's equations, *Lecture Notes in Comp. Sci. Engl.*, **28** Springer, Berlin, 2003.
[8] P. Monk, *Finite Element Methods for Maxwell Equations*, Oxford University Press, Oxford, 2003.
[9] J. C. Nédélec, Mixed finite elements in $\mathbb{R}^3$. *Numer. Math.*, **35**(1980), 315-341.
[10] J. C. Nédélec, A new family of mixed finite elements in $\mathbb{R}^3$. *Numer. Math.*, **50** (1986), 47-81.