

ALGEBRAIC MULTIGRID FOR MARKOV CHAINS

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Abstract. An algebraic multigrid (AMG) method is presented for the calculation of the stationary probability vector of an irreducible Markov chain. A modified AMG interpolation formula is proposed that produces a nonnegative interpolation operator with unit row sums. It is shown how the adoption of a lumping technique maintains the irreducible singular M-matrix character of the coarse-level operators on all levels. Together, these properties are sufficient to guarantee the well-posedness of the algorithm. Numerical results show how it leads to nearly optimal multigrid efficiency for a representative set of test problems.

Key words. multilevel method, Markov chain, stationary probability vector, algebraic multigrid

AMS subject classifications. 65C40 Computational Markov chains, 60J22 Computational methods in Markov chains, 65F10 Iterative methods for linear systems, 65F15 Eigenvalues, eigenvectors

1. Introduction. This paper describes an algebraic multigrid (AMG) method for computing the stationary probability vector of large, sparse, irreducible Markov transition matrices.

While multigrid methods, including multilevel aggregation methods [10, 8] and smoothed aggregation [7], have been considered before for Markov chains, our present approach is based on AMG for nonsingular linear systems. At the same time, it also incorporates some aspects of recent work on aggregation multigrid for Markov chains.

Starting from the classical definition of AMG interpolation described in [5], we propose a modified interpolation formula, which produces a nonnegative interpolation operator with unit row sums. Furthermore, it is shown how the adoption of a lumping technique, which was recently employed in an aggregation-based method for Markov chains [7], maintains the irreducible singular M-matrix character of the coarse-level operators on all levels. We show numerically that the resulting lumped AMG method for Markov chains (MCAMG) leads to nearly optimal multigrid efficiency for a representative set of test problems for which traditional iterative methods are slow to converge. The aim here is to use a sequence of successively coarser versions of the original problem to remedy the slow convergence that plagues traditional one-level iterative methods, like the power method, when the subdominant eigenvalue satisfies $|\lambda_2| \approx 1$ [13].

The use of AMG has already been explored for Markov chain problems [17], in the context of additive AMG used as a preconditioner for GMRES. Our formulation, however, is multiplicative, and near-optimal results are obtained without needing GMRES acceleration. Our formulation is also different in that it is related to adaptive AMG [4].

Large sparse Markov chains are of interest in a wide range of applications, including information retrieval and web ranking, performance modelling of computer and communication systems, dependability and security analysis, and analysis of biological systems [15]. Multilevel solvers for Markov problems with improved efficiency thus promise to have significant impact in many disciplines.

2. Mathematical formulation. The problem of finding the stationary probability vector of a Markov chain can be stated as follows. Given a column-stochastic transition-probability matrix, $B \in \mathbb{R}^{n \times n}$, i.e., $0 \leq b_{ij} \leq 1$ and

$$\mathbf{1}^T B = \mathbf{1}^T, \tag{2.1}$$

we seek the vector $\mathbf{x} \in \mathbb{R}^n$, such that

$$B \mathbf{x} = \mathbf{x}, \quad x_i \geq 0 \quad \forall i, \quad \|\mathbf{x}\|_1 = 1. \tag{2.2}$$

Here, $\mathbf{1}$ is the column vector of all ones. It can be shown that, if B is *irreducible*, then there exists a unique solution to (2.2) with strictly positive components. This is a consequence of the Perron-Frobenius theorem for nonnegative matrices [9]. In what follows, we consider the case where B is irreducible. For the sake of completeness, we formally define irreducibility here.

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DEFINITION 2.1 (Directed walk and directed path).

For nodes u and v in a directed graph, $D = (\mathcal{N}, \mathcal{A})$, with node set \mathcal{N} and arc set \mathcal{A} , a u - v walk in D is a finite sequence of nodes $u = v_0, v_1, \dots, v_{k-1}, v_k = v$, beginning at u and ending at v , such that $(v_{i-1}, v_i) \in \mathcal{A}$ for $i = 1, \dots, k$. A directed u - v path is a directed u - v walk in which no node is repeated.

DEFINITION 2.2 (Directed graph of a matrix).

The directed graph of $A \in \mathbb{R}^{n \times n}$, denoted by $\Gamma(A)$, is the directed graph on n nodes v_1, \dots, v_n such that an arc exists from v_i to v_j if and only if $a_{ji} \neq 0$.

DEFINITION 2.3 (Irreducible matrix).

Matrix $A \in \mathbb{R}^{n \times n}$ is called irreducible if and only if there exists a directed path from v_i to v_j for any two distinct nodes $v_i, v_j \in \mathcal{N}(\Gamma(A))$.

3. Singular M-matrices. Following the approach outlined in [17], we can, equivalently, restate the problem of solving for the stationary probability vector as solving for a strictly positive vector of unit length in the nullspace of a singular M-matrix. Mathematically, we seek the vector $\mathbf{x} \in \mathbb{R}^n$ such that

$$A\mathbf{x} = \mathbf{0}, \quad x_i > 0 \quad \forall i, \quad \|\mathbf{x}\|_1 = 1, \quad (3.1)$$

where $A := I - B$ and B is irreducible. Here, A is a singular M-matrix and $\mathbf{1}^T A = \mathbf{0}$.

We now define singular M-matrices, show that A belongs to this class, and state a number of properties shared by all singular M-matrices. These properties, together with irreducibility, provide a solid theoretical foundation with which we can prove the well-posedness of our algorithm. Let $\rho(B)$ be the spectral radius of B . Then, we have the following definition:

DEFINITION 3.1 (Singular M-matrix).

$A \in \mathbb{R}^{n \times n}$ is a singular M-matrix if and only if there exists $B \in \mathbb{R}^{n \times n}$ with $b_{ij} \geq 0$ for all i, j , such that $A = \rho(B)I - B$.

The justification that $A := I - B$ is a singular M-matrix now follows readily from Definition 3.1 and from the fact that $\rho(B) = 1$ for any column-stochastic matrix B . Furthermore, it is easy to see that, if B is irreducible, then A must also be irreducible, since subtracting B from I cannot zero out any off-diagonal elements of B (refer to Definition 2.3). The following properties of singular M-matrices are taken from [2, 7, 17]. We choose to restate them here so that the paper is self-contained.

THEOREM 3.2 (Some properties of singular M-matrices).

- (1) Irreducible singular M-matrices yield a unique solution to $A\mathbf{x} = \mathbf{0}$, up to scaling, which can be chosen such that all components of \mathbf{x} are strictly positive.
- (2) Irreducible singular M-matrices have nonpositive off-diagonal elements and strictly positive diagonal elements ($n > 1$).
- (3) If A has a strictly positive vector in its left or right nullspace and its off-diagonal elements are nonpositive, then A is a singular M-matrix.
- (4) If A is an irreducible singular M-matrix, then each of its principal submatrices, other than A itself, is a nonsingular M-matrix (Theorem 4.16 in [2]).

As we shall see, property (2) allows us to construct an interpolation operator with nonnegative entries and unit row sums. These properties are also used below to prove that the coarse operators of our AMG method are irreducible singular M-matrices on all levels.

4. Algebraic multigrid for Markov Chains. In this section, we recall the principal features of the classical AMG V-cycle [3, 5, 16] on which our method is based. We discuss how our approach for Markov chains deviates from the classical approach for nonsingular linear systems, and how it incorporates aspects of recent work on aggregation multigrid for Markov chains [7]. We conclude this section by describing our V-cycle algorithm and proving well-posedness.

4.1. Multiplicative AMG method and coarsening. One major difference between our approach for Markov chains and that of classical AMG for nonsingular linear systems is the use of multiplicative error, \mathbf{e}_i , defined by $\mathbf{x} = \text{diag}(\mathbf{x}_i) \mathbf{e}_i$, where \mathbf{x} is the exact solution of (3.1) and \mathbf{x}_i is the i th iterate. As we will see below, the iterates \mathbf{x}_i obtained by our algorithm have strictly positive components. Equation (3.1) can then be rewritten as

$$A \text{diag}(\mathbf{x}_i) \mathbf{e}_i = \mathbf{0}. \quad (4.1)$$

We observe that, at convergence, $\mathbf{x}_i = \mathbf{x}$ and, hence, $\mathbf{e}_i = \mathbf{1}$. This motivates the following definition of the multiplicative coarse-level error correction:

$$\mathbf{x}_{i+1} = \text{diag}(\mathbf{x}_i) P \mathbf{e}_c, \quad (4.2)$$

where P is the interpolation operator (see Section 4.2) and \mathbf{e}_c is the error approximation on the coarse level. It is easy to see that (4.2) is the natural extension of the additive coarse-level error correction to the multiplicative case.

Now consider the scaled fine-level operator $\bar{A} := A \text{diag}(\mathbf{x}_i)$. By Theorem 3.2(3), \bar{A} is also an irreducible singular M-matrix. We can rewrite Equation (4.1) in terms of \bar{A} , which results in the following fine-level error equation:

$$\bar{A} \mathbf{e}_i = \mathbf{0}. \quad (4.3)$$

To seek a coarse representation of Equation (4.3), we first perform the two-pass AMG coarsening routine described in [5], which determines the set of points on the coarse level. Basing *strength of connection* (and coarsening) on the scaled operator, \bar{A} , is motivated heuristically in [8]. Recall that, given a threshold value, $\theta \in [0, 1]$, the point i depends strongly on the point j if

$$-\bar{a}_{ij} \geq \theta \max_{k \neq i} \{-\bar{a}_{ik}\}. \quad (4.4)$$

In this paper we use $\theta = 0.25$. At convergence, $\mathbf{1}$ lies in the nullspace of \bar{A} , so standard AMG coarsening and interpolation approaches work well. The coarse-level version of (4.3) is given by

$$R \bar{A} P \mathbf{e}_c = \mathbf{0} \quad \text{or} \quad \bar{A}_c \mathbf{e}_c = \mathbf{0}, \quad (4.5)$$

with $\bar{A}_c := R \bar{A} P$ and the restriction operator defined by the variational property, $R = P^T$.

The attentive reader may question the well-posedness of the coarse-level equation. In fact, the coarse-level operator \bar{A}_c may not be an irreducible singular M-matrix. As we show in Section 4.3, this problem is remedied by applying a *lumping method* [7] to the coarse-level operator, whereby we obtain the *lumped* coarse-level operator, \hat{A}_c . We prove below that this lumped \bar{A}_c is an irreducible singular M-matrix, and that the exact solution, \mathbf{x} , is a fixed point of the V-cycle with the lumped coarse-level error equation, $\hat{A}_c \mathbf{e}_c = \mathbf{0}$.

We conclude this section by stating two identities for the unlumped \bar{A}_c , which are used in Section 4.3. Let the coarse-level column vector of all ones be denoted by $\mathbf{1}_c$. We choose P such that $P \mathbf{1}_c = \mathbf{1}$ (see below), which implies that

$$\mathbf{1}_c^T \bar{A}_c = \mathbf{0} \quad \forall \mathbf{x}_i, \quad (4.6)$$

$$\bar{A}_c \mathbf{1}_c = \mathbf{0} \quad \text{for } \mathbf{x}_i = \mathbf{x}. \quad (4.7)$$

4.2. Interpolation. The interpolation operator, P , transfers information from coarse to fine levels. It is constructed in such a way that it accurately represents fine-level *algebraically smooth components*, by which we mean components whose error is not effectively reduced by relaxation. In the AMG method, interpolation is accomplished by approximating the error at each fine-level point (F -point) as a weighted sum of the error at coarse-level points (C -points). In what follows, we recall the definition of the AMG interpolation operator from [5], and explain how the formula for the interpolation weights is modified to obtain the properties for P that are desirable for Markov chain problems.

Suppose we have already performed coarsening on the current set of fine-level points, $H = \{1, \dots, m\}$, and have, thus, partitioned H into a set of coarse (C) and fine (F) points. Then, for any point $i \in H = C \cup F$, we require that

$$(P \mathbf{e}_c)_i = \begin{cases} (\mathbf{e}_c)_i & \text{if } i \in C, \\ \sum_{j \in C_i} w_{ij} (\mathbf{e}_c)_j & \text{if } i \in F, \end{cases} \quad (4.8)$$

where \mathbf{e}_c is the coarse-level error approximation, w_{ij} are the interpolation weights, and C_i is the set of C -points that strongly influence point i according to (4.4). Observe that, for any $i \in C$, row i of

P is all zeros except for the entry corresponding to C -point i , which equals 1. A classical formula for AMG interpolation weights is derived in [5]:

$$w_{ij} = -\frac{\bar{a}_{ij} + \sum_{m \in D_i^s} \left(\frac{\bar{a}_{im}\bar{a}_{mj}}{\sum_{k \in C_i} \bar{a}_{mk}} \right)}{\bar{a}_{ii} + \sum_{r \in D_i^w} \bar{a}_{ir}}, \quad (4.9)$$

where $C_i \cup D_i^s \cup D_i^w = N_i$, the directed neighborhood of point i , which is the set of all points $k \neq i$ such that $\bar{a}_{ik} \neq 0$. Here, D_i^s is the set of F -points that strongly influence i , and D_i^w is the set of points in the neighborhood N_i that do not strongly influence i . Note that D_i^w may contain both F -points and C -points.

We cannot directly use this formula for our problem for several reasons: the denominator in (4.9) may be zero, some of the weights may be negative, and the weights do not sum to one. We do indeed desire an interpolation operator whose rows sum to unity, that is, we desire a P such that $P \mathbf{1}_c = \mathbf{1}$. This is a necessary condition to establish identities (4.6) and (4.7). Note that this also implies that constants are interpolated exactly. To ensure that P enjoys this property, we rescale the w_{ij} s of (4.8) with the rescaled weights \bar{w}_{ij} given by

$$\bar{w}_{ij} = \frac{\bar{a}_{ij} + \sum_{m \in D_i^s} \left(\frac{\bar{a}_{im}\bar{a}_{mj}}{\sum_{k \in C_i} \bar{a}_{mk}} \right)}{\sum_{p \in C_i} \bar{a}_{ip} + \sum_{r \in D_i^s} \bar{a}_{ir}}. \quad (4.10)$$

Under the premise that \bar{A} is an irreducible singular M-matrix, Theorem 3.2(2) ensures that all matrix elements in (4.10) are negative. Since the two-pass AMG coarsening routine ensures that $C_i \neq \emptyset$, it follows that the denominator in Equation (4.10) is nonzero. Furthermore, together with the fact that C_i , D_i^s and D_i^w do not have points in common, we find that $\bar{w}_{ij} > 0$ for all $i \in F$ and $j \in C_i$. Thus, the redefined interpolation operator has nonnegative entries and unit row sums. Note that it is important to perform both passes of the coarsening routine, since this ensures that $\sum_{k \in C_i} \bar{a}_{mk} \neq 0$ for any $i \in F$ and $m \in D_i^s$, which is required for the w_{ij} s to be well-defined. It is the second pass of the coarsening routine that ensures that every point in D_i^s strongly depends on at least one point in C_i .

4.3. Lumping. As we mentioned at the close of Section 4.1, the coarse-level operator, \bar{A}_c , may not be an irreducible singular M-matrix. To illustrate this point, we let matrices D , L and U be such that $\bar{A} = D - (L + U)$, where D is diagonal, L is strictly lower triangular, and U is strictly upper triangular. Then

$$\bar{A}_c = P^T \bar{A} P = P^T D P - P^T (L + U) P = S - G, \quad (4.11)$$

where both $S = P^T D P$ and $G = P^T (L + U) P$ are nonnegative matrices because \bar{A} is a singular M-matrix and P has nonnegative entries. $P^T D P$ is generally not diagonal, so \bar{A}_c may have positive off-diagonal entries, meaning it may not be a singular M-matrix. Furthermore, \bar{A}_c may lose irreducibility due to new zero entries being introduced. To rectify this problem, we adopt the lumping method described in [7] for smoothed aggregation multigrid methods for Markov chains. An overview of the lumping procedure is provided below.

We consider a modified version, \hat{S} , of S , which is obtained by *lumping* parts of S to the diagonal (explained below), resulting in the modified coarse-level operator

$$\hat{A}_c = \hat{S} - G. \quad (4.12)$$

Our goal is to modify S in such a way that \hat{A}_c has nonpositive off-diagonal elements and retains nonzero off-diagonal elements where G has them (to guarantee irreducibility).

Define an *offending index pair* as a tuple (i, j) such that $i \neq j$ and $s_{ij} \neq 0$ and $(\bar{A}_c)_{ij} \geq 0$. It is for these indices that lumping is performed. Let (i, j) be an offending index pair. To correct for the sign in \bar{A}_c at location (i, j) we add a matrix $S_{\{i,j\}}$ to S , with the elements of $S_{\{i,j\}}$ equaling $\beta_{\{i,j\}}$ at positions

(i, i) and (j, j) , $-\beta_{\{i,j\}}$ at positions (i, j) and (j, i) , and otherwise being zero. This corresponds to lumping parts of S to the diagonal, in the sense that $\beta_{\{i,j\}}$ is removed from off-diagonal elements s_{ij} and s_{ji} and added to diagonal elements s_{ii} and s_{jj} . We choose $\beta_{\{i,j\}} > 0$ so that $s_{ij} - g_{ij} - \beta_{\{i,j\}} < 0$ and $s_{ji} - g_{ji} - \beta_{\{i,j\}} < 0$, resulting in strictly negative off-diagonal elements in \hat{A}_c at locations (i, j) and (j, i) . Note that $\beta_{\{i,j\}}$ is chosen such that adding $S_{\{i,j\}}$ for correcting the sign at location (i, j) also corrects the sign at location (j, i) , if necessary. This means that, if both (i, j) and (j, i) are offending index pairs, then only one matrix $S_{\{i,j\}}$ has to be added to S . In our implementation,

$$\beta_{\{i,j\}} = \max\{s_{ij} + (\eta - 1)g_{ij}, s_{ji} + (\eta - 1)g_{ji}\}, \quad (4.13)$$

and η is a fixed parameter $\in (0, 1]$. It is important to note that while lumping may introduce new nonzero entries into \hat{A}_c , it cannot create a zero entry in \hat{A}_c where G is nonzero. Finally, we experimentally observed that we should lump as little as possible, so η should be chosen small [7]. In practice, $\eta = 0.01$ seems to be a good choice.

Finally, symmetric matrices such as $S_{\{i,j\}}$ are used to modify S so that column sums and row sums of \bar{A}_c are conserved. This ensures that properties (4.6) and (4.7) are retained after lumping: \hat{A}_c has $\mathbf{1}$ as a left-kernel vector on all levels and, at convergence, has $\mathbf{1}$ as a right-kernel vector. Indeed, since $(\hat{S} - S) = \sum S_{\{i,j\}}$, where the sum is over all matrices $S_{\{i,j\}}$ added to S , it follows that

$$\mathbf{1}_c^T \hat{A}_c = \mathbf{1}_c^T \bar{A}_c + \mathbf{1}_c^T (\hat{S} - S) = \mathbf{1}_c^T \bar{A}_c = \mathbf{0} \quad \forall \mathbf{x}_i, \quad (4.14)$$

$$\hat{A}_c \mathbf{1}_c = \bar{A}_c \mathbf{1}_c + (\hat{S} - S) \mathbf{1}_c = \bar{A}_c \mathbf{1}_c = \mathbf{0} \quad \text{for } \mathbf{x}_i = \mathbf{x}. \quad (4.15)$$

4.4. Relaxation. This paper uses weighted Jacobi for all relaxation operations. Decomposing matrix A into its diagonal and negative strictly upper and lower triangular parts, $A = D - (L + U)$, weighted Jacobi for solving $A\mathbf{x} = \mathbf{0}$ is given by

$$\mathbf{x}_{i+1} = (1 - \omega)\mathbf{x}_i + \omega D^{-1}(L + U)\mathbf{x}_i \quad (4.16)$$

where $\omega \in (0, 1)$ is a fixed weight parameter. We observe that if A is an irreducible singular M-matrix, then Theorem 3.2 confirms that D^{-1} exists, and that $D^{-1}(L + U)$ has nonnegative entries. Thus, $\mathbf{x}^{(k+1)}$ has strictly positive entries if $\mathbf{x}^{(k)}$ has strictly positive entries and $\omega \in (0, 1)$. Since we can normalize the result after relaxation, the constraint that \mathbf{x} be a probability vector is easily obtained.

4.5. MCAMG V-cycle algorithm. Now that our approach has been described, we state our V-cycle algorithm for Markov chains:

Algorithm 1: MCAMG($A, \mathbf{x}, \nu_1, \nu_2$), AMG for Markov chains (V-cycle)

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if not at the coarsest level then
   $\mathbf{x} \leftarrow \text{Relax}(A, \mathbf{x}) \nu_1$  times
   $\bar{A} \leftarrow A \text{diag}(\mathbf{x})$ 
  Compute the set of coarse-level points  $C$ 
  Construct the interpolation operator  $P$ 
  Construct the coarse-level operator  $\bar{A}_c \leftarrow P^T \bar{A} P$ 
  Obtain the lumped coarse-level operator  $\hat{A}_c \leftarrow \text{Lump}(\bar{A}_c, \eta)$ 
   $\mathbf{e}_c \leftarrow \text{MCAMG}(\hat{A}_c, \mathbf{1}_c, \nu_1, \nu_2)$  /* coarse-level solve */
   $\mathbf{x} \leftarrow \text{diag}(\mathbf{x}) P \mathbf{e}_c$  /* coarse-level correction */
   $\mathbf{x} \leftarrow \text{Relax}(A, \mathbf{x}) \nu_2$  times
else
   $\mathbf{x} \leftarrow$  direct solve of  $A\mathbf{x} = \mathbf{0}$ 
end

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Note that the set of coarse-level points, C , and the interpolation operator, P , are recalculated for each V-cycle on each level. In principle, however, the sets of coarse-level points and the interpolation operators can be “frozen” after a few cycles to reduce the amount of work, but this is not done for the results presented in this paper.

4.6. Well-posedness of MCAMG. We require well-posedness of this algorithm in the sense that, given an iterate that is strictly positive, the algorithm gives a proper definition for the next iterate. We begin by proving the following proposition, which is the key result necessary to prove irreducibility of \hat{A}_c .

PROPOSITION 4.1 (irreducibility of G).

If $\bar{A} = D - (L + U)$ is an irreducible singular M-matrix, then $G = P^T(L + U)P$ is irreducible.

Proof. We need to show that, for any C -points with coarse-level labels I and J , there exists a directed path from node I to node J in the directed graph of G . First, observe that if \bar{A} is irreducible, then $(L + U)$ is irreducible, since diagonal entries do not matter for irreducibility. Assume that $(L + U)_{kl} \neq 0$ for some fine-level labels k and l and let I be any C -point that interpolates to l , that is, $p_{lI} \neq 0$. Similarly, let J be any C -point that interpolates to k , that is, $p_{kJ} \neq 0$. In Section 4.2, we showed that every row of P contains at least one nonzero element; hence, indices I and J exist. Now,

$$g_{IJ} = \mathbf{p}_I^T(L + U)\mathbf{p}_J,$$

where \mathbf{p}_I denotes column I of P and \mathbf{p}_J denotes column J of P . Since both $(\mathbf{p}_I)_l$ and $(\mathbf{p}_J)_k$ are nonzero and $(L + U)_{kl} \neq 0$, it follows by the nonnegativity of P and $(L + U)$ that $g_{IJ} \neq 0$. Thus, for any fine-level points l and k such that there exists an arc from node l to node k in $\Gamma(\bar{A})$, there must also exist coarse-level points I and J such that there is an arc from node I to node J in $\Gamma(G)$.

Now, let I and J be any distinct C -points. Furthermore, let i and j be the fine-level labels of I and J , respectively. By the irreducibility of $(L + U)$, there exists a directed path of distinct fine-level points from node i to node j . Denote this path by

$$i = v_0, v_1, \dots, v_{k-1}, v_k = j,$$

where the nodes v_0, \dots, v_k are fine-level points. By the result above, there must exist coarse-level points V_0, \dots, V_k that form the directed walk (see Definition 2.1)

$$V_0, V_1, \dots, V_{k-1}, V_k$$

in $\Gamma(G)$. However, any directed U - V walk contains a directed U - V path [6]. Thus, we can find a directed path in $\Gamma(G)$ that begins at V_0 and ends at V_k . Finally, we observe that C -points V_0 and V_k were chosen such that they interpolate to i and j , respectively. But, by the definition of P , the only point that interpolates to a given C -point is the point itself. Thus, it follows that $V_0 = I$ and $V_k = J$. Therefore, there exists a directed path from node I to node J in the directed graph of G . Since I and J were arbitrary, G is irreducible. \square

Well-posedness of the algorithm now follows from the first of the following two theorems; the second theorem is a requirement for convergence of the method.

THEOREM 4.2 (Singular M-matrix property of lumped coarse-level operator).

\hat{A}_c is an irreducible singular M-matrix on all coarse levels and, thus, has a unique right-kernel vector with positive components (up to scaling) on all levels.

Proof. Assume that \bar{A} is an irreducible singular M-matrix and let $\bar{A} = D - (L + U)$. By Proposition 4.1, matrix $G = P^T(L + U)P$ is irreducible. Lumping ensures that \hat{A}_c has nonzero entries where G has nonzero entries. Hence, \hat{A}_c is irreducible. To establish the singular M-matrix property, observe that lumping ensures that \hat{A}_c has nonpositive off-diagonal entries. It follows by (4.14) and Theorem 3.2(3) that \hat{A}_c is an irreducible singular M-matrix. By Theorem 3.2(1), \hat{A}_c has a unique right-kernel vector with strictly positive components (up to scaling). The proof now follows formally by induction over the levels. \square

THEOREM 4.3 (Fixed-point property of MCAMG V-cycle).

The exact solution, \mathbf{x} , is a fixed point of the MCAMG V-cycle.

Proof. Property (4.15) implies that $\mathbf{e}_c = \mathbf{1}_c$ is a solution of the coarse-level equation $\hat{A}_c \mathbf{e}_c = \mathbf{0}$ for $\mathbf{x}_i = \mathbf{x}$. We note that this solution is unique (up to scaling) since \hat{A}_c is an irreducible singular M-matrix. The coarse-level correction formula then gives $\mathbf{x}_{i+1} = \text{diag}(\mathbf{x}_i)P\mathbf{e}_c = \text{diag}(\mathbf{x})P\mathbf{1}_c = \mathbf{x}$. The result now follows by the fact that the exact solution, \mathbf{x} , is a fixed point of the weighted Jacobi relaxation scheme (see Section 4.4). \square

5. Numerical results. In this section we present numerical convergence results for MCAMG. Testing is performed for a variety of problems which fall into two distinct categories: those for which B has a real spectrum, and those for which the spectrum of B is complex. In the latter case we plot the spectrum of B for a given problem size.

In the tables that follow, n is the number of degrees of freedom and γ is the geometric mean of the convergence factors of the last five V-cycles, which are defined as the ratios of the one-norm of the residual, $\|A \mathbf{x}_i\|_1$, after and before each cycle. Note that the \mathbf{x}_i are scaled such that $\|\mathbf{x}_i\|_1 = 1$. For all the numerical results presented in this paper, we start from a random, strictly positive initial guess and iterate until the residual has been reduced by a factor of 10^{-8} measured in the one-norm, or until 100 cycles have been performed, whichever comes first. We perform a direct solve on the coarse level when $n < 12$. All V-cycles used are (1,1) cycles, with one pre-relaxation and one post-relaxation on each level. A scalable (or optimal) method requires γ to be uniformly bounded away from one as n is increased, resulting in the number of required iterations to be bounded as well. In the tables, it is the number of iterations performed and lev is the number of levels in the last cycle. The weight in the weighted Jacobi relaxation is chosen as $\omega = 0.7$. The operator complexity of the last cycle, C_{op} , is defined as the sum of the number of nonzero elements in all operators, A , on all levels divided by the number of nonzero elements in the fine-level operator. This number gives a good indication of the amount of work required for a cycle and, for a scalable (or optimal) method, it should be bounded by a constant not too much larger than one as n increases. We also provide an effective convergence factor, defined as $\gamma_{eff} = \gamma^{1/C_{op}}$. This effective convergence factor takes work into account and makes it easier to evaluate the overall efficiency of the method as n increases. For a scalable method, γ_{eff} should be bounded uniformly by a constant less than one as the problem size increases. Finally, R_l is the lumping ratio of the last cycle, defined as the sum of the number of ‘‘offending’’ elements in operators A on all levels divided by the sum of the number of nonzero elements in A on all levels. This ratio gives the fraction of matrix elements for which lumping is required, and is, thus, an indication of the extra work required for lumping. Note that no lumping is required in the fine-level matrix, so lumping only contributes extra work starting from the second level.

For the first four test problems, we compare our results with numeric tests performed using Algebraic Smoothed Aggregation for Markov chains (A-SAM) in [7]. Depending on the case, so-called distance-one or distance-two aggregation are employed (see [7]), whichever is the most efficient.

5.1. Real spectrum problems. In this section we consider test problems for which B has a real spectrum. These include a uniform two-dimensional (2D) lattice, an anisotropic 2D lattice and a random walk on an unstructured planar graph. Each test problem has also been considered in [7], so our description is brief. The test problems are generated by undirected graphs with weighted edges. The weights determine the transition probabilities: the transition probability from node i to j is given by the weight of the edge from node i to j , divided by the sum of the weights of all outgoing edges from node i . It is easy to show that the spectrum of the resulting transition matrices is real.

In the uniform lattice, all weights are chosen equal to 1, and in the anisotropic lattice, horizontal weights are 1 while vertical weights are 10^{-6} (see [7]). The numerical results for the uniform lattice and the anisotropic lattice are given in Table 5.1 and Table 5.2, respectively. The results obtained

n	MCAMG						A-SAM [7] distance-two					
	γ	it	C_{op}	γ_{eff}	lev	R_{lump}	γ	it	C_{op}	γ_{eff}	lev	R_{lump}
64	0.22	11	2.02	0.47	3	0	0.42	18	1.25	0.50	2	0.0e-0
256	0.25	12	2.20	0.54	5	0	0.47	19	1.35	0.57	3	1.2e-3
1024	0.23	11	2.20	0.51	6	0	0.49	20	1.42	0.60	4	4.5e-3
4096	0.23	11	2.20	0.52	7	0	0.49	20	1.47	0.62	4	1.7e-3
16384	0.24	11	2.20	0.52	8	0	0.59	20	1.56	0.72	5	1.4e-3
65536	0.24	11	2.20	0.52	9	0	0.66	21	1.59	0.77	6	1.3e-3

TABLE 5.1
Uniform 2D lattice.

for the uniform lattice and the anisotropic lattice are very similar and, in both cases, the MCAMG V-cycles lead to computational complexity that is optimal. Furthermore, we observe that, in each

n	MCAMG						A-SAM [7] distance-two					
	γ	it	C_{op}	γ_{eff}	lev	R_{lump}	γ	it	C_{op}	γ_{eff}	lev	R_{lump}
64	0.19	11	2.15	0.46	4	0	0.40	17	1.76	0.59	3	0.0e-0
256	0.19	11	2.42	0.51	6	0	0.33	15	2.23	0.61	4	7.4e-4
1024	0.18	11	2.58	0.52	8	0	0.33	14	2.81	0.68	5	1.6e-3
4096	0.18	11	2.67	0.53	10	0	0.33	14	3.43	0.73	7	4.9e-4
16384	0.18	11	2.73	0.54	12	0	0.33	13	4.17	0.77	7	2.5e-4
65536	0.18	11	2.76	0.54	14	0	0.32	13	4.80	0.79	9	7.6e-5

TABLE 5.2
Anisotropic 2D lattice ($\varepsilon = 1e - 6$).

case, the same number of iterations are required to achieve convergence and the effective convergence factors are almost identical. Note also that no lumping is required on the last cycle. In both cases, it is clear that MCAMG significantly outperforms A-SAM [7].

In our final test problem of this section, we consider an unstructured planar (undirected) graph and calculate the stationary probability distribution of the random walk on the graph. The graph is generated by choosing n random points in the unit square, and triangulating them using Delaunay triangulation. The random walk on the graph is modelled by a Markov chain, with the transition probability from node i to node j given by the reciprocal of the number of edges incident on node i (equal weights). Table 5.3 shows good convergence results for the unstructured planar graph problem

n	MCAMG						A-SAM [7] distance-one					
	γ	it	C_{op}	γ_{eff}	lev	R_{lump}	γ	it	C_{op}	γ_{eff}	lev	R_{lump}
1024	0.40	15	2.13	0.65	6	0	0.53	20	1.69	0.68	5	2.6e-2
2048	0.33	14	2.22	0.61	7	3.2e-5	0.52	19	1.68	0.68	5	2.1e-2
4096	0.40	15	2.19	0.66	7	3.2e-5	0.61	21	1.80	0.76	5	2.4e-2
8192	0.40	15	2.25	0.66	8	4.7e-5	0.64	22	1.92	0.79	7	2.5e-2
16384	0.37	14	2.26	0.65	9	3.5e-5	0.76	30	2.03	0.87	7	2.4e-2
32768	0.37	14	2.28	0.65	9	7.6e-5	0.74	28	2.08	0.86	7	2.4e-2

TABLE 5.3
Unstructured planar graph.

with very little lumping on the last cycle. It appears that C_{op} is bounded, and consideration of γ and the number of iterations suggest that the computational complexity is optimal. Compared to the results in [7], it is evident that MCAMG significantly outperforms A-SAM for this test problem.

5.2. Complex spectrum problems. In this section we consider the test problems for which B has a complex spectrum. These include a tandem queueing network and a stochastic Petri net problem. These test problems have also been considered in [7, 10, 15]. We conclude this section with a plot of the spectrum of B for each test problem.

The first test problem is an open tandem queueing network from [15]; see also [7]. Table 5.4 shows the numerical results for the tandem queueing network test problem. Iteration numbers are constant

n	MCAMG						A-SAM [7] distance-two					
	γ	it	C_{op}	γ_{eff}	lev	R_{lump}	γ	it	C_{op}	γ_{eff}	lev	R_{lump}
256	0.32	15	4.08	0.75	5	7.6e-2	0.39	18	1.94	0.61	4	1.1e-1
1024	0.32	15	4.41	0.77	7	7.4e-2	0.41	20	2.04	0.64	4	7.6e-2
4096	0.32	16	4.49	0.78	8	7.0e-2	0.45	24	2.12	0.69	5	5.5e-2
16384	0.32	16	4.58	0.78	10	9.2e-2	0.56	30	2.18	0.77	6	5.3e-2
65536	0.33	15	4.56	0.78	11	4.9e-2	0.71	37	2.37	0.86	6	1.3e-1

TABLE 5.4
Tandem queueing network.

and the operator complexity grows somewhat as a function of problem size, but appears bounded. The amount of lumping required for this nonsymmetric 2D problem is larger than for the previous problems, but is still small and does not add much extra work. These results are competitive with those obtained using A-SAM in [7].

The final test problem we consider is derived from a stochastic Petri net (SPN). Petri nets are a formalism for the description of concurrency and synchronization in distributed systems. They consist of: *places*, which model conditions or objects; *tokens*, which represent the specific value of the condition or object; *transitions*, which model activities that change the value of conditions or objects; and *arcs*, which specify interconnection between places and transitions. A stochastic Petri net is a standard Petri net, together with a tuple $\Lambda = (r_1, \dots, r_n)$ of exponentially distributed transition firing rates. Furthermore, we know from [11] that a finite place, finite transition, marked stochastic Petri net is isomorphic to a one-dimensional discrete space Markov process. For an in-depth discussion of Petri Nets, the reader is referred to [1, 11].

We test MCAMG on the SPN described in [10]. Table 5.5 shows that the operator complexity is

n	γ	it	C_{op}	γ_{eff}	lev	R_{lump}
1496	0.38	16	4.72	0.82	9	2.28e-2
2470	0.38	16	4.86	0.82	10	2.28e-2
3795	0.38	15	5.17	0.83	10	2.16e-2
10416	0.39	16	5.71	0.85	11	2.09e-2
16206	0.39	16	5.89	0.85	11	2.07e-2
23821	0.39	16	6.11	0.86	12	2.08e-2
33511	0.39	16	6.29	0.86	13	2.08e-2
45526	0.39	16	6.47	0.87	13	2.09e-2

TABLE 5.5
Stochastic Petri net, $\Lambda = (1, 3, 7, 9, 5)$.

fairly large for this example and grows somewhat as the problem size increases. However, we observe that the effective convergence factor, γ_{eff} , is essentially constant, so optimal complexity is nearly achieved.

Figure 5.1 shows the unit circle in the complex plane and the spectrum of B for each test problem.

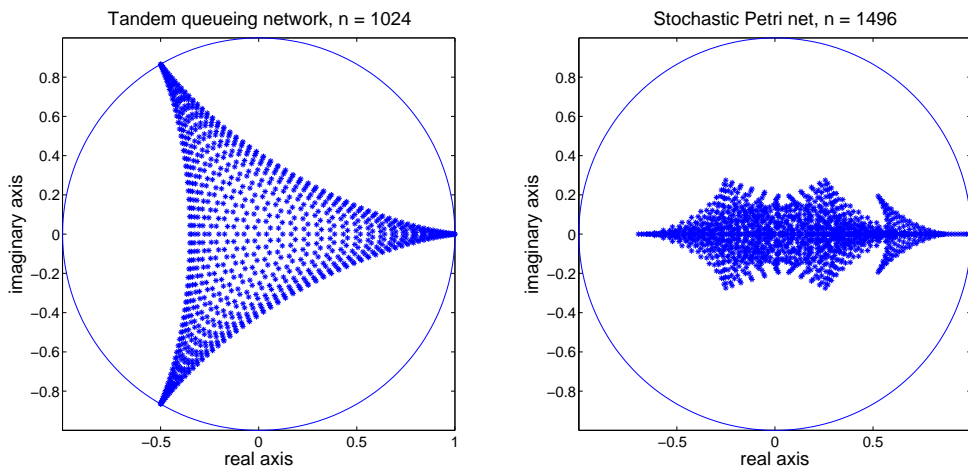


FIG. 5.1. Complex spectra of B .

6. Conclusions and future work. We demonstrated how multiplicative algebraic multigrid with lumping and a modified interpolation formula can be used to find the stationary probability vector of a Markov chain. It was shown that, with lumping and a modified interpolation formula, the coarse-level operators are irreducible singular M-matrices on all levels, resulting in strictly positive coarse-level corrections. It was also shown that the exact solution is a fixed point of our algorithm. We performed numerical testing for a wide variety of test problems, and considered problems with both real and complex spectra. For each test case, we observed that MCAMG V-cycles lead to optimal or nearly optimal multigrid efficiency, that is, the effective convergence factor, γ_{eff} , was bounded uniformly by a constant less than one as the problem size increased. We observed that MCAMG was competitive with Algebraic Smoothed Aggregation Multigrid for Markov chains (A-SAM) from [7], and in most cases, outperformed A-SAM.

Further possible avenues of research include parallel implementations of MCAMG, acceleration of AMG V-cycle convergence using combinations of previous iterates as in Krylov methods [14], and use of acceleration on all recursive levels as in K-cycle methods [12].

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