MULTIGRID METHODS FOR SYMMETRIC DISCONTINUOUS GALERKIN METHODS ON GRADED MESHES

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ABSTRACT. We present theoretical and numerical results for multigrid methods for several discontinuous Galerkin methods on graded meshes.

1. INTRODUCTION

In this article we present theoretical and numerical results for multigrid methods for a class of symmetric, consistent and stable discontinuous Galerkin methods on graded meshes. These methods are suitable for domains with re-entrant corners. Details can be found in [9, 8].

For simplicity we restrict to a simple model problem. Let $\Omega \in \mathbb{R}^2$ be a bounded polygonal domain. Denote the corners of Ω by $c_1,\ c_2, \cdots, c_L$ and interior angles at these corners by $\omega_1,\ \omega_2, \cdots, \omega_L$. We associate the parameters $\mu_1,\ \mu_2, \cdots, \mu_L$ to $c_1,\ c_2, \cdots, c_L$ by

$$\begin{cases} \mu_\ell = 1 & \omega_\ell < \pi \\ \dfrac{1}{2} < \mu_\ell < \dfrac{\pi}{\omega_\ell} & \omega_\ell > \pi \end{cases}, \tag{1.1}$$

and define the weight function ϕ_μ by $\phi_\mu(x) = \prod_{\ell=1}^{L} |x - c_\ell|^{1-\mu_\ell}$. We then define the weighted Sobolev space

$$L_{2,\mu}(\Omega) = \{f \in L_{2,\mathrm{loc}}(\Omega) :\ \|f\|^2_{L_{2,\mu}(\Omega)} = \int_\Omega \phi_\mu^2(x) f^2(x)\, dx < \infty\}.$$

The model problem is to find $u \in H^1_0(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx \qquad \forall\ v \in H^1_0(\Omega), \tag{1.2}$$

where f belong to the weighted Sobolev space $L_{2,\mu}(\Omega)$.

Sobolev's inequality implies that $\int_\Omega |fv|\, dx \le C_\Omega \|f\|_{L_{2,\mu}(\Omega)} \|v\|_{H^1(\Omega)}$ for all $v \in H^1(\Omega)$. Hence the model problem (1.2) has a unique solution u for any $f \in L_{2,\mu}(\Omega)$. Moreover u has the following properties.

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(i) The second order weak derivatives of u belong to $L_{2,\mu}$ and they satisfy

$$\|\partial^2 u/\partial x_i \partial x_j\|^2_{L_{2,\mu}(\Omega)} = \int_\Omega \phi_\mu^2(x)(\partial^2 u/\partial x_i \partial x_j)^2(x)\,dx \le C_\Omega \|f\|^2_{L_{2,\mu}(\Omega)}$$

for $1 \le i, j \le 2$.

(ii) Let $\delta > 0$ be small enough so that the neighborhoods $\Omega_{\ell,\delta} = \{x \in \Omega : |x - c_\ell| < \delta\}$ around the corners c_ℓ for $1 \le \ell \le L$ are disjoint. At a reentrant corner c_ℓ where $\omega_\ell > \pi$, we have $u \in H^{1+\mu_\ell}(\Omega_{\ell,\delta})$ and

$$\|u\|_{H^{1+\mu_\ell}(\Omega_{\ell,\delta})} \le C_\Omega \|f\|_{L_{2,\mu}(\Omega)}.$$

(iii) u is continuous on $\bar{\Omega}$.

Details can be found for example in [13, 12, 16].

2. DG methods on graded meshes

To recover optimal convergence rates for domains with re-entrant corners where the solution of the model problem does not belong to $H^2(\Omega)$, we use a triangulation $\mathcal{T}_h$ of Ω satisfying

$$\Phi_\mu(T)h \approx h_T \qquad \forall\, T \in \mathcal{T}_h, \tag{2.1}$$

where $h_T = \operatorname{diam} T$ and $h = \max_{T\in\mathcal{T}_h} h_T$. Here the weight $\Phi_\mu(T)$ is defined by

$$\Phi_\mu(T) = \prod_{\ell=1}^{L} |c_\ell - c_T|^{1-\mu_\ell},$$

where the grading parameters $\mu_1, \ldots, \mu_L$ are chosen according to (1.1) and c_T is the center of T.

We define the P_1 discontinuous finite element space V_h by

$$V_h = \{v \in L_2(\Omega) : v_T = v\big|_T \in P_1(T) \quad \forall\, T \in \mathcal{T}_h\}.$$

Denote by $\mathcal{E}_h$ (resp. $\mathcal{E}_h^I$) the set of all edges (resp. interior edges) of $\mathcal{T}_h$. Let $v \in V_h$. We define the mean $\{\{\nabla v\}\}$, and jump $[[v]]$ as follows. Let $e \in \mathcal{E}_h^I$ be an interior edge shared by two triangles $T_{e,1}, T_{e,2} \in \mathcal{T}_h$ and $\boldsymbol{n}_1$ (resp. $\boldsymbol{n}_2$) be the unit normal of e pointing towards the outside of $T_{e,1}$ (resp. $T_{e,2}$). We define, on e,

$$\{\{\nabla v\}\} = \frac{1}{2}(\nabla v_{T_{e,1}|e} + \nabla v_{T_{e,2}|e}) \quad \text{and} \quad [[v]] = v_{T_{e,1}|e}\boldsymbol{n}_1 + v_{T_{e,2}|e}\boldsymbol{n}_2.$$

Let $e \in \mathcal{E}_h$ be a boundary edge. We take $\boldsymbol{n}_e$ to be the unite normal of e pointing towards the outside of Ω and define

$$\{\{\nabla v\}\} = \nabla v|_e \quad \text{and} \quad [[v]] = v|_e \boldsymbol{n}_e.$$

Given $f \in L_{2,\mu}(\Omega)$, we consider four discontinuous Galerkin methods for (1.2): Find $u_h \in V_h$ such that

$$a_h(u_h, v) = \int_\Omega f v \, dx \qquad \forall\, v \in V_h, \tag{2.2}$$

where

$$\begin{aligned} a_h(w, v) = \sum_{T \in \mathcal{T}_h} \int_T \nabla w \cdot \nabla v \, dx - \big(\sum_{e \in \mathcal{E}_h} \int_e \{\!\{\nabla w\}\!\} \cdot [\![v]\!] \, ds + \sum_{e \in \mathcal{E}_h} \int_e \{\!\{\nabla v\}\!\} \cdot [\![w]\!] \, ds \big) \\ + \theta \int_\Omega \ell_h\big([\![w]\!]\big) \cdot \ell_h\big([\![v]\!]\big) \, dx + Q_h(w, v), \end{aligned} \tag{2.3}$$

with $\theta = 1$ or 0, $Q_h = Q^j$ or Q^r, Q^j defined by

$$Q^j(w, v) = \eta \sum_{e \in \mathcal{E}_h} \frac{1}{|e|} \int_e [\![w]\!] \cdot [\![v]\!] \, ds, \tag{2.4}$$

Q^r defined by

$$Q^r(w, v) = \eta \sum_{e \in \mathcal{E}_h} \int_\Omega \ell_e\big([\![w]\!]\big) \cdot \ell_e\big([\![v]\!]\big) \, ds, \tag{2.5}$$

and $\eta > 0$ a penalty parameter. In (2.3) and (2.5), the local lifting operator $\ell_e : [L^2(e)]^2 \to V_h \times V_h$ and the global lifting operator $\ell_h : [L^2(\mathcal{E}_h)]^2 \to V_h \times V_h$ are defined by

$$\begin{aligned} \int_\Omega \ell_e(\varphi) \cdot w \, dx &= - \int_e \varphi \cdot \{\!\{w\}\!\} \, ds && \forall\, w \in V_h \times V_h,\ \varphi \in [L^2(e)]^2, \\ \int_\Omega \ell_h(\varphi) \cdot w \, dx &= - \sum_{e \in \mathcal{E}_h} \int_e \varphi \cdot \{\!\{w\}\!\} \, ds && \forall\, w \in V_h \times V_h,\ \varphi \in [L^2(\mathcal{E}_h)]^2. \end{aligned}$$

The classification of these four methods is given in Table 2.1, where η_* is a sufficiently large positive number.

TABLE 2.1. DG methods

Method	θ	Q_h	η
Brezzi et al. [10]	1	Q^r	$\eta > 0$
LDG [11]	1	Q^j	$\eta > 0$
Bassi et al. [4]	0	Q^r	$\eta > 3$
IP [1]	0	Q^j	$\eta > \eta^*$

These DG methods are consistent and they are also stable under the conditions on η given in Table 2.1 (cf. [2]). Consequently, we have the quasi-optimal error estimate

$$\|u - u_h\|_h \le C \inf_{v \in V_h} \|u - v\|_h, \tag{2.6}$$

where the energy norm of v is defined by

$$\|v\|_h^2 = \sum_{T\in\mathcal{T}_h} \|\nabla v\|_{L_2(T)}^2 + \sum_{e\in\mathcal{E}_h} |e|\, \|\{\!\{\nabla v\}\!\}\|_{L_2(e)}^2 + \sum_{e\in\mathcal{E}_h} \frac{1}{|e|} \|[\![v]\!]\|_{L_2(e)}^2$$

and the constant C in (2.6) depends only on the minimum angle of $\mathcal{T}_h$.

To turn the abstract error estimate (2.6) into a concrete estimate, we need an interpolation operator. Let $\Pi_h : C(\bar{\Omega}) \longrightarrow V_h$ be the nodal interpolation operator for the conforming P_1 finite element, i.e., $\Pi_h u \in V_h \cap H^1(\Omega)$ agrees with u at the vertices of the triangles of $\mathcal{T}_h$. The following lemma provides an interpolation error estimate for Π_h, whose proof (cf. [9]) uses the properties (i)–(iii) of u stated in Section 1.

Lemma 2.1. *Let $f \in L_{2,\mu}(\Omega)$ and $u \in H_0^1(\Omega)$ satisfy* (1.2). *Then*

$$\|u - \Pi_h u\|_h \le Ch\|f\|_{L_{2,\mu}(\Omega)}. \tag{2.7}$$

Combining (2.6), (2.7) and a standard duality argument, we obtain the following theorem that indicates these DG methods have optimal convergence rates on domains with re-entrant corners.

Theorem 2.2. *Let $f \in L_{2,\mu}(\Omega)$, u be the solution of* (1.2), *and u_h be the solution of any of the DG methods in Table* 2.1 *associated with a triangulation $\mathcal{T}_h$ that satisfies* (2.1). *We have the following error estimate*:

$$\|u - u_h\|_{L_2(\Omega)} + h\|u - u_h\|_h \le Ch^2\|f\|_{L_{2,\mu}(\Omega)}. \tag{2.8}$$

3. Multigrid Methods

Starting with an initial triangulation $\mathcal{T}_0$, we construct a family of nested triangulations of Ω that satisfies (2.1). For a given level k, we divide each triangle $T \in \mathcal{T}_k$ into four triangles according to the following rules to obtain $\mathcal{T}_{k+1}$.

(i) If none of the reentrant corners is a vertex of T, we divide T uniformly by connecting the midpoints of the edges of T.

(ii) If a reentrant corner c_ℓ is a vertex of T and the other two vertices of T are denoted by p_1 and p_2, then we divide T by connecting the points m_ℓ, m_1 and m_2 (cf. Figure 3.1). Here m_ℓ is the midpoint of the edge p_1p_2 and m_1 (resp. m_2) is the point on the edge $c_\ell p_1$ (resp. $c_\ell p_2$) such that

$$\Big|\frac{c_\ell - m_i}{c_\ell - p_i}\Big| = 2^{-(1/\mu_\ell)} \qquad \text{for} \quad i = 1, 2,$$

where μ_ℓ is the grading factor chosen according to (1.1).

The refinement procedure is identical with the one in [5]. The triangulations $\mathcal{T}_0$, $\mathcal{T}_1$ and $\mathcal{T}_2$ for an L-shaped domain are depicted in Figure 3.2, where the grading factor at the reentrant corner is taken to be 2/3.

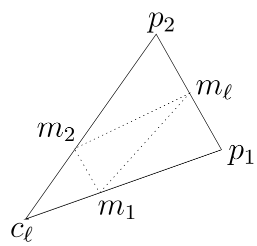

FIGURE 3.1. Refinement of a triangle at a reentrant corner

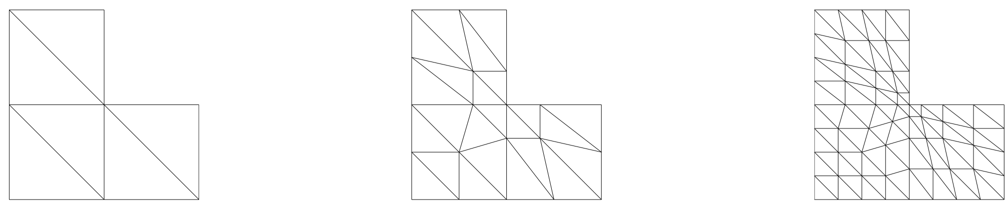

FIGURE 3.2. The triangulations $\mathcal{T}_0$, $\mathcal{T}_1$ and $\mathcal{T}_2$ for an L-shaped domain

Let V_k be the discontinuous P_1 finite element space associated with $\mathcal{T}_k$ and $a_k(\cdot,\cdot)$ be the analog of $a_h(\cdot,\cdot)$. We can rewrite (2.2) on $\mathcal{T}_k$ as

$$A_k u_k = f_k, \tag{3.1}$$

where $A_k : V_k \longrightarrow V_k'$ and $f_k \in V_k'$ are defined by

$$\langle A_k w, v\rangle = a_k(w,v) \quad \forall\, v, w \in V_k \quad \text{and} \quad \langle f_k, v\rangle = \int_\Omega f v\, dx \quad \forall\, v \in V_k. \tag{3.2}$$

Here $\langle\cdot,\cdot\rangle$ is the canonical bilinear form on $V_k' \times V_k$. Equations of the form (3.1) can be solved by V-cycle, W-cycle and F-cycle multigrid algorithms [14, 15, 17].

There are two key ingredients in the design of a multigrid algorithm. We need intergrid transfer operators to move functions between grids and a good smoother to damp out the highly oscillatory part of the error. Since the finite element spaces are nested, we can take the coarse-to-fine intergrid transfer operator $I_{k-1}^k : V_{k-1} \longrightarrow V_k$ to be the natural injection and define the fine-to-coarse intergrid transfer operator $I_k^{k-1} : V_k' \longrightarrow V_{k-1}'$ to be the transpose of I_{k-1}^k with respect to the canonical bilinear forms, i.e.,

$$\langle I_k^{k-1}\alpha, v\rangle = \langle \alpha, I_{k-1}^k v\rangle \qquad \forall\, \alpha \in V_k',\ v \in V_{k-1}.$$

In order to define the smoother, we first introduce an operator $B_k : V_k \longrightarrow V_k'$ defined by

$$\langle B_k w, v\rangle = \sum_{T\in\mathcal{T}_k} \sum_{m\in\mathcal{M}_T} w(m)v(m) \qquad \forall\, v, w \in V_k, \tag{3.3}$$

where $\mathcal{M}_T$ is the set of the midpoints of the three edges of T. It is easy to see from (2.3), (3.2), and (3.3) that we can choose a (constant) damping factor λ so that the spectral radius

$\rho(\lambda B_k^{-1}A_k)$ satisfies

$$\rho(\lambda B_k^{-1}A_k) < 1 \quad \text{for} \quad k \geq 0. \tag{3.4}$$

We will use the following preconditioned Richardson relaxation scheme for the equation $A_k z = g$ as the smoother:

$$z_{\text{new}} = z_{\text{old}} + \lambda B_k^{-1}(g - A_k z_{\text{old}}).$$

The convergence of the W-cycle algorithm is discussed in the next section and the numerical results for V-cycle, W-cycle and F-cycle algorithms are reported in Section 5.

4. Convergence Analysis for the W-cycle Multigrid Algorithm

The convergence analysis for W-cycle algorithm follows the ideas in [3, 18], with modifications provided by [6, 7] that can overcome the difficulty that, for nonconforming methods, the energy norm is no longer preserved by the coarse-to-fine intergrid transfer operator I_{k-1}^k.

Let $E_k : V_k \longrightarrow V_k$ be the error propagation operator for the k-th level W-cycle algorithm with m_1 pre-smoothing and m_2 post-smoothing steps. We have the following well-known recursive relation [14]:

$$E_k = R_k^{m_2}(Id_k - I_{k-1}^k P_k^{k-1} + I_{k-1}^k E_{k-1}^2 P_k^{k-1})R_k^{m_1}, \tag{4.1}$$

where Id_k is the identity operator on V_k, the operator $R_k : V_k \longrightarrow V_k$ which measures the effect of one smoothing step is defined by

$$R_k = Id_k - \lambda B_k^{-1}A_k, \tag{4.2}$$

and the operator $P_k^{k-1} : V_k \longrightarrow V_{k-1}$ is the transpose of I_{k-1}^k with respect to the variational forms, i.e.,

$$a_{k-1}(P_k^{k-1}w, v) = a_k(w, I_{k-1}^k v) \qquad \forall\, v \in V_{k-1},\ w \in V_k.$$

Let the mesh-dependent norms $|\!|\!| v |\!|\!|_{j,k}$ for $j = 0, 1, 2$ and $k \geq 1$ be defined by

$$|\!|\!| v |\!|\!|_{j,k} = \sqrt{\langle B_k(B_k^{-1}A_k)^j v, v\rangle} \qquad \forall\, v \in V_k,\ k \geq 1. \tag{4.3}$$

In particular, we have

$$|\!|\!| v |\!|\!|_{0,k}^2 = \langle B_k v, v\rangle \quad \text{and} \quad |\!|\!| v |\!|\!|_{1,k}^2 = \langle A_k v, v\rangle \quad \forall\, v \in V_k.$$

Also the Cauchy-Schwarz inequality implies that

$$|\!|\!| v |\!|\!|_{2,k} = \max_{w \in V_k \setminus \{0\}} \frac{\langle A_k v, w\rangle}{|\!|\!| w |\!|\!|_{0,k}} \qquad \forall\, v \in V_k.$$

The smoothing properties in the following lemma are simple consequences of (3.4), (4.2) and (4.3). Their proofs are standard [14].

Lemma 4.1. *There exists a positive constant C independent of k such that*

$$|\!|\!| R_k v |\!|\!|_{1,k} \leq |\!|\!| v |\!|\!|_{1,k},\ |\!|\!| R_k^m v |\!|\!|_{1,k} \leq C(1+m)^{-1/2} |\!|\!| v |\!|\!|_{0,k} \text{ and } |\!|\!| R_k^m v |\!|\!|_{2,k} \leq C(1+m)^{-1/2} |\!|\!| v |\!|\!|_{1,k}$$

for all $v \in V_k$ and $k \geq 1$.

The approximation property in the next lemma is proved by using the Galerkin orthogonality of the DG methods, the interpolation error estimate (2.7), the relation

$$\langle B_k v, v\rangle \approx h_k^{-2}\int_\Omega \phi_\mu^{-2}(x)v^2(x)\,dx \qquad \forall\, v \in V_k,$$

and a duality argument.

Lemma 4.2. *There exists a positive constant C independent of k such that*

$$|||(Id_k - I_{k-1}^k P_k^{k-1})v|||_{0,k} \le C|||v|||_{2,k} \qquad \forall\, v \in V_k,\ k \ge 1.$$

Combining Lemma 4.1, Lemma 4.2 and a perturbation argument, we obtain the following convergence result for the W-cycle algorithm.

Theorem 4.3. *There exists a positive constant C and a positive integer m_* independent of k such that*

$$|||E_k v|||_{1,k} \le C[(1+m_1)(1+m_2)]^{-1/2}|||v|||_{1,k} \qquad \forall\, v \in V_k,\ k \ge 0, \tag{4.4}$$

provided $m_1 + m_2 \ge m_*$.

5. Numerical Results

We computed the contraction numbers of the W-cycle, F-cycle and V-cycle algorithms for the model problem (1.2) on the L-shaped domain $(-1,1)^2 \setminus ([0,1]\times[0,-1])$. The initial triangulation $\mathcal{T}_0$ has six elements (cf. Figure 3.2). The triangulations $\mathcal{T}_1,\ldots,\mathcal{T}_7$ are created by the refinement procedure described at the beginning of Section 3. We used $\eta = 1$ (resp. $\eta = 10$) as the penalty parameter, $\lambda = 1/20$ (resp. $\lambda = 1/40$) as the damping factor, and m pre-smoothing and m post-smoothing steps for the LDG (resp. IP) method. The results are presented in Tables 5.1–5.6. It is observed that for both methods the W-cycle and the F-cycle have similar contraction numbers. The contraction numbers of the method by Brezzi et al. (resp. Bassi et al.), which are not reported here, are larger than the corresponding contraction numbers for the LDG (resp. IP) method.

	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
$m=3$	0.99	0.88	0.65	0.63	0.63	0.63	0.64
$m=4$	0.65	0.43	0.51	0.55	0.57	0.56	0.57
$m=5$	0.43	0.38	0.45	0.49	0.50	0.51	0.52
$m=6$	0.28	0.33	0.41	0.44	0.465	0.47	0.47
$m=7$	0.18	0.29	0.37	0.39	0.42	0.43	0.44
$m=8$	0.13	0.26	0.33	0.37	0.39	0.39	0.40
$m=9$	0.11	0.24	0.30	0.32	0.36	0.37	0.38

Table 5.1. Contraction numbers of the W-cycle algorithm on the L-shaped domain using LDG

	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
$m=4$	0.65	0.43	0.51	0.54	0.57	0.57	0.58
$m=5$	0.43	0.38	0.45	0.49	0.50	0.52	0.52
$m=6$	0.28	0.33	0.41	0.44	0.45	0.47	0.47
$m=7$	0.18	0.29	0.37	0.39	0.42	0.43	0.44
$m=8$	0.13	0.26	0.33	0.37	0.39	0.39	0.40
$m=9$	0.11	0.24	0.30	0.32	0.36	0.37	0.38
$m=10$	0.09	0.22	0.29	0.32	0.35	0.36	0.36

Table 5.2. Contraction numbers of the F-cycle algorithm on the L-shaped domain using LDG

	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
$m=5$	0.43	0.45	0.53	0.64	0.72	0.77	0.81
$m=6$	0.28	0.35	0.36	0.42	0.48	0.48	0.50
$m=7$	0.18	0.31	0.38	0.41	0.48	0.48	0.49
$m=8$	0.13	0.28	0.34	0.39	0.41	0.45	0.46
$m=9$	0.11	0.25	0.30	0.35	0.39	0.43	0.43
$m=10$	0.09	0.23	0.28	0.35	0.38	0.40	0.41
$m=11$	0.07	0.21	0.28	0.33	0.37	0.38	0.39

Table 5.3. Contraction numbers of the V-cycle algorithm on the L-shaped domain using LDG

	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
$m=2$	0.82	0.77	0.80	0.79	0.79	0.80	0.79
$m=3$	0.61	0.71	0.71	0.73	0.75	0.75	0.76
$m=4$	0.47	0.61	0.68	0.72	0.72	0.73	0.73
$m=5$	0.44	0.59	0.66	0.68	0.70	0.70	0.71
$m=6$	0.39	0.54	0.61	0.66	0.67	0.69	0.69
$m=7$	0.35	0.52	0.59	0.65	0.66	0.67	0.67
$m=8$	0.31	0.48	0.59	0.64	0.65	0.66	0.66

Table 5.4. Contraction numbers of the W-cycle algorithm on the L-shaped domain using IP

The asymptotic behavior of the contraction numbers of the W-cycle, and V-cycle algorithms for LDG and IP methods with respect to the number of smoothing steps for $k=6$ is depicted in Figure 5.1. The log-log graphs confirm that the contraction numbers decrease at the rate of m^{-1}. We also observed that the contraction numbers for Bassi et al. and Brezzi et al. have similar behaviors.

	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
$m=4$	0.47	0.61	0.68	0.72	0.72	0.73	0.73
$m=5$	0.44	0.59	0.66	0.68	0.70	0.70	0.71
$m=6$	0.39	0.54	0.61	0.66	0.67	0.69	0.69
$m=7$	0.35	0.52	0.59	0.65	0.66	0.68	0.68
$m=8$	0.31	0.48	0.59	0.64	0.65	0.66	0.66
$m=9$	0.28	0.47	0.58	0.62	0.64	0.65	0.65
$m=10$	0.25	0.45	0.56	0.62	0.63	0.64	0.64

TABLE 5.5. Contraction numbers of the F-cycle algorithm on the L-shaped domain with using IP

	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
$m=6$	0.39	0.56	0.60	0.69	0.71	0.73	0.77
$m=7$	0.35	0.53	0.57	0.69	0.71	0.72	0.73
$m=8$	0.31	0.49	0.62	0.67	0.69	0.70	0.71
$m=9$	0.28	0.47	0.60	0.64	0.67	0.69	0.70
$m=10$	0.25	0.45	0.56	0.64	0.67	0.68	0.69
$m=11$	0.24	0.44	0.55	0.64	0.66	0.67	0.68
$m=12$	0.22	0.42	0.54	0.61	0.64	0.67	0.67

TABLE 5.6. Contraction numbers of the V-cycle algorithm on the L-shaped domain using IP

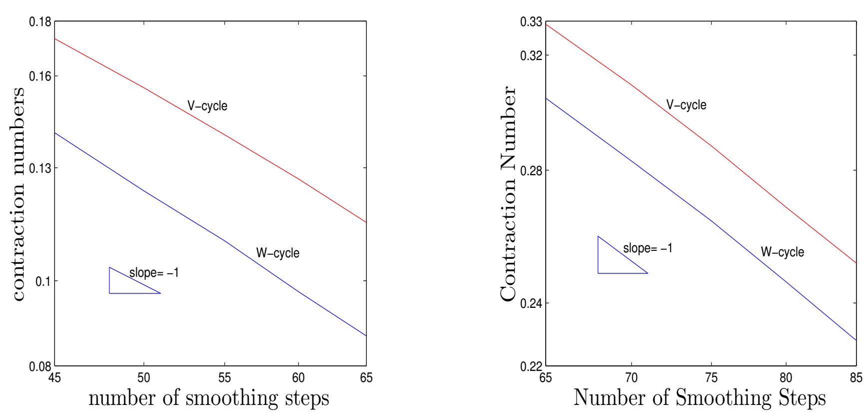

FIGURE 5.1. Asymptotic behavior of the contraction number using LDG (left) and IP (right) with respect to the number of smoothing steps

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