ON THE ACCURACY OF MULTIGRID TRUNCATION ERROR ESTIMATES ON STAGGERED GRIDS

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Abstract. Multigrid methods can provide computable estimates of the truncation error for a discretized partial differential equation by comparing discretizations on grids of two different mesh sizes. [11] studied the differences between the standard formulation and a more accurate formulation for linear problems limited on non-staggered grids. This paper extends the accurate truncation error estimates to non-linear problems on staggered grids and gives applications to the Cauchy-Riemann equations, Stokes equations and Navier-Stokes equations.

Key words. multigrid, truncation error, tau extrapolation, Cauchy-Riemann equations, Navier-Stokes equations

AMS subject classifications. 65N15, 65N50, 65N55

1. Introduction.

Multigrid methods use approximations on grids of different mesh sizes to obtain fast solvers for boundary value problems [2, 8]. Comparing the approximations on different grids also provides computable estimates of the truncation error on the coarse grid, which can then be used in adaptive grid refinement algorithms or in extrapolation to higher-order accuracy (τ-extrapolation). The basic procedures have been presented in many places, e.g., [6, 8, 14]. However, the standard formulation (e.g., [14]) includes an assumption (not always explicitly stated) on the residual transfer operator; when this assumption is violated, the truncation error estimates are inaccurate unless a high-order residual transfer is used. A more general formulation [11] based on the approach of [12] which gives accurate truncation error estimates in all cases without the need for high-order residual transfers. However, the analysis in [11] was limited to linear differential operators on non-staggered grids.

The purpose of this paper is to extend the analysis to nonlinear partial differential equations on staggered grids. Bernert [1] has discussed τ-extrapolation for Navier-Stokes equations on staggered grids. According to [1], the only way to do τ-extrapolation (and presumably to compute accurate truncation error estimates) on staggered grids is to use high-order restrictions from fine to coarse grid and high-order averaging operators on the coarse grid. However, based on the analysis and numerical results presented here, a much simpler approach exists. The analysis follows [11] by using *pointwise* asymptotic estimates (rather than bounds on error norms). The presentation is organized as follows. Section 2 defines the problem and notation. In section 3 we state and prove the main result relating the accuracy of the truncation error estimates to the accuracy of the discretization and grid transfers. Numerical results for the Cauchy-Riemann equations are given in section 4 and applications to the Stokes equations are written in section 5. The results for the nonlinear Navier-Stokes equations are displayed in section 6. Discussion and conclusions are summarized in section 7.

2. Problem formulation.

A nonlinear partial differential equation is denoted by

$$\mathcal{L}(u) = 0 \tag{2.1}$$

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on a domain $\Omega \subset \mathbb{R}^d$, where $\mathcal{L} : C^n(\Omega) \to C(\Omega)$ is a nonlinear differential operator and $u \in C^n(\Omega)$ is the solution. The boundary condition is ignored here since it has no impact on the analysis presented in this paper. Note that a linear PDE $Lu = f$ is a special case of (2.1) with $\mathcal{L}(u) := Lu - f$. If $\mathcal{L}$ is C^2, and if $u + tw$, $0 \le t \le 1$, is contained in $C^n(\Omega)$, by using the Taylor formula, we have

$$\mathcal{L}(u + w) = \mathcal{L}(u) + \mathbf{D}\mathcal{L}(w) + O|w|^2, \tag{2.2}$$

where $\mathbf{D}\mathcal{L}$ is the Fréchet derivative [4, 5].

The discretization of Equation (2.1) on a grid Ω^h indexed by a mesh size parameter h is of the form

$$\mathcal{L}^h(u^h) = 0, \tag{2.3}$$

where $\mathcal{L}^h$ is the discrete form of the operator on grid Ω^h, and u^h is the corresponding (exact) solution of the discrete equation. While we have in mind finite-difference discretizations, this formulation could also describe finite element or other discretizations. Similarly, for the discrete equation, if $\mathcal{L}^h$ is C^2, and if $u + tw$, $0 \le t \le 1$, is $\in C^n(\Omega)$, we have

$$\mathcal{L}^h(\hat{I}^h u + \hat{I}^h w) = \mathcal{L}^h(\hat{I}^h u) + \mathbf{D}\mathcal{L}^h(\hat{I}^h w) + O||\hat{I}^h w||^2, \tag{2.4}$$

where $\mathbf{D}\mathcal{L}^h$ is Fréchet derivative [3] and $\hat{I}^h$ represents *linear restriction* (e.g., pointwise restriction) from Ω to grid Ω^h.

The corresponding (local) truncation error is

$$\tau^h = \tau^h(u) := \mathcal{L}^h(\hat{I}^h u). \tag{2.5}$$

Comparing (2.2) with (2.4), when the truncation error is order of p, and $w = O(h^p)$, by applying the Fréchet derivative for the truncation error term of order p, there exists the approximation

$$\mathbf{D}\mathcal{L}^h(\hat{I}^h w) = \hat{I}^h \mathbf{D}\mathcal{L}(w) + O(h^{p+p}), \tag{2.6}$$

In addition to the discrete equation (2.3), a multigrid method also uses a corresponding discrete equation on a coarser grid Ω^H, with mesh ratio $\rho = h/H < 1$ (usually $\rho = 1/2$). The coarse grid problem is denoted as

$$\mathcal{L}^H(u^H) = 0. \tag{2.7}$$

Corresponding to the linear *relative local truncation error* in [11]

$$\tau_h^H := L^H(\hat{I}_h^H \tilde{u}^h) - f^H - I_h^H(L^h \tilde{u}^h - f^h), \tag{2.8}$$

we define the nonlinear *relative local truncation error*

$$\tau_h^H := \mathcal{L}^H(\hat{I}_h^H \tilde{u}^h) - I_h^H \mathcal{L}^h(\tilde{u}^h). \tag{2.9}$$

Here $\tilde{u}^h$ is the current approximation to the true (discrete) solution u^h of the fine-grid equation (2.3), $\hat{I}_h^H$ represents the fine-to-coarse transfer of the solution (which is not necessary the same as the solution transfer used FAS because there is no connection between these two solution restriction operators), and I_h^H is the fine-to-coarse residual transfer operator (e.g., full weighting).

3. Analysis. Our main result is the following.

THEOREM 3.1 (Truncation Error Estimate). *Assume that there exists $p \geq 1$ and $q \geq 1$ such that if $u \in C^{n+p+q}(\Omega)$, and $\mathcal{L}$ is C^2, the truncation error (2.5) satisfies*
(A1) $\tau^h = h^p \hat{I}^h v + O(h^{p+q})$ *with* $v \in C^{n+q}(\Omega)$,
that the approximate solution $\tilde{u}^h$ of the discrete problem (2.3) satisfies
(A2) $\tilde{u}^h = \hat{I}^h(u + w)$, $w = O(h^p)$,
and that there exists $r \geq 1$ and $t \geq 1$such that for any $\phi \in C^r(\Omega)$, and $\psi \in C^t(\Omega)$,
(A3) $I_h^H \hat{I}^h \phi = \hat{I}^H \phi + O(h^r)$
(A4) $\hat{I}_h^H \hat{I}^h \psi = \hat{I}^H \psi + O(h^t)$.
Then

$$\tau^H - \gamma \tau_h^H = O(h^\alpha) \tag{3.1}$$

where $\gamma = (1 - \rho^p)^{-1} = H^p/(H^p - h^p)$ and $\alpha = \min(p + q, t, p + r, p + p)$.

Proof. Following [11] we use (A1) and (A3) to estimate the truncation error difference between grids Ω^h and Ω^H as

$$\begin{aligned}(\Delta\tau)_h^H &:= \tau^H - I_h^H \tau^h \\ &= H^p \hat{I}^H v - h^p I_h^H \hat{I}^h v + O(h^{p+q}) \\ &= H^p (1 - \rho^p) \hat{I}^H v + O(h^{p+r}) + O(h^{p+q}) \\ &= (1 - \rho^p)\, \tau^H + O(h^{p+r}) + O(h^{p+q}).\end{aligned} \tag{3.2}$$

We then use (A2), (A3), (A4), (2.4), and (2.6) relate $(\Delta\tau)_h^H$ to τ_h^H via

$$\begin{aligned}\tau_h^H - (\Delta\tau)_h^H &= \left[\mathcal{L}^H(\hat{I}_h^H \tilde{u}^h) - I_h^H \mathcal{L}^h(\tilde{u}^h)\right] - \left[\mathcal{L}^H(\hat{I}^H u) - I_h^H \mathcal{L}^h(\hat{I}^h u)\right] \\ &= \left[\mathcal{L}^H\left(\hat{I}_h^H \hat{I}^h (u + w)\right) - I_h^H \mathcal{L}^h\left(\hat{I}^h (u + w)\right)\right] - \left[\mathcal{L}^H(\hat{I}^H u) - I_h^H \mathcal{L}^h(\hat{I}^h u)\right] \\ &= \mathbf{D}\mathcal{L}^H(\hat{I}^H w) - I_h^H \mathbf{D}\mathcal{L}^h(\hat{I}^h w) + O(h^t) + O(h^{p+p}) \\ &= \hat{I}^H \mathbf{D}\mathcal{L}(w) - I_h^H \hat{I}^h \mathbf{D}\mathcal{L}(w) + O(h^t) + O(h^{p+p}) \\ &= O(h^{p+r}) + O(h^t) + O(h^{p+p}).\end{aligned} \tag{3.3}$$

Combining (3.2) and (3.3) yields the desired result. ∎

Similarly to the linear case, a higher-order extrapolation is obtained by choosing $t \geq 3$ to calculate the truncation error estimate:

COROLLARY 3.2 (τ-Extrapolation). *Under the assumptions of Theorem 3.1, the extrapolated discretization*

$$\mathcal{L}^H \bar{u}^h = \bar{f}^H := \gamma \tau_h^H \tag{3.4}$$

has accuracy $O(h^\alpha)$.

Proof. Using (3.1) the associated truncation error is

$$\bar{\tau}^H := \mathcal{L}^H \hat{I}^H u - \bar{f}^H = \tau^H - \gamma \tau_h^H = O(h^\alpha). \qquad \square \tag{3.5}$$

As noted by Schaffer [12] and Hackbusch [16], at convergence on grid Ω^h, the extrapolated equation (3.4) gives the same result as Richardson extrapolation for u^h when $\mathcal{L}^h(u^h) := L^h u^h - f^h$ is affine.

4. Cauchy-Riemann equations. As a first example of solving a system of partial differential equations on staggered grid consider the Cauchy-Riemann equations

$$u_x + v_y = F_1, \tag{4.1a}$$
$$u_y - v_x = F_2, \tag{4.1b}$$

on a domain $\Omega = [0,1] \times [0,1]$ with Dirichlet boundary conditions: specifying the true solution value of u on east and west boundary and the true solution value of v on north and south boundary, where $u = u(x,y)$ and $v = v(x,y)$ are the unknown functions, the subscripts denote partial derivative, and F_1 and F_2 are given functions of x and y.

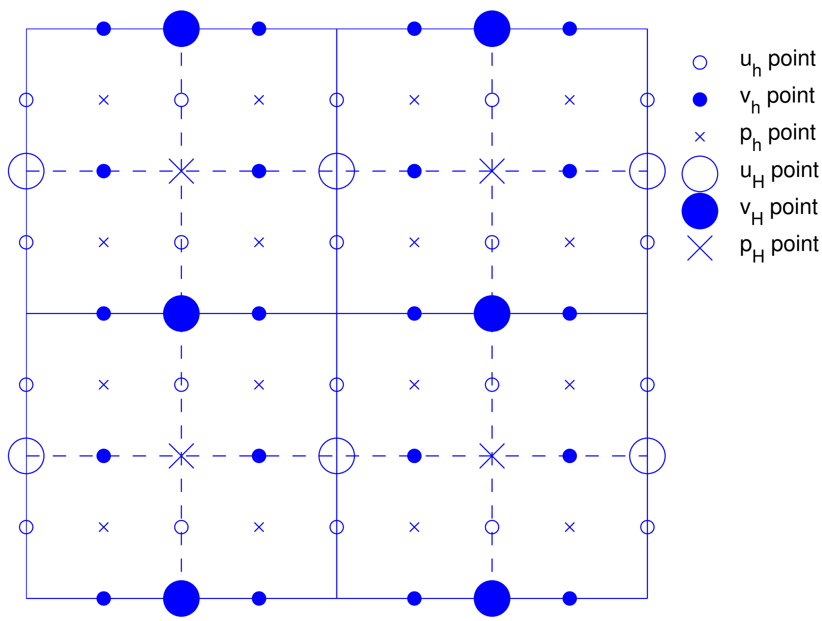

FIG. 4.1. *Sixteen fine and four coarse staggered grid cells and corresponding unknowns. For the Cauchy-Reimann equations, the first equation is defined at the cell centers and second equation is defined at cell vertices. For the Stokes and Navier-Stokes equations, the first momentum equation is defined at u points, the second momentum is defined at v points, and the third continuity equation is defined at the cell centers. This figure is after [15]).*

We approximate equations (4.1) by central difference on staggered grid (shown in Fig. 4.1 in the following form

$$\partial_x^h u^h + \partial_y^h v^h = F_1(x,y) \qquad \text{at cell centers,} \tag{4.2a}$$
$$\partial_y^h u^h - \partial_x^h v^h = F_2(x,y) \qquad \text{at interior vertices,} \tag{4.2b}$$

where $\partial_x^h \phi^h = \frac{1}{h}(\phi(x+\frac{h}{2},y) - \phi(x-\frac{h}{2},y))$ and ∂_y^h is defined similarly. The truncation error is $O(h^2)$, and $(A1)$ holds with $p = 2$ and $q = 2$.

A multigrid method with Distributive Gauss-Seidel (DGS) relaxation ([6, 18]) as its smoother on each grid is used to solve this system. The coarse grid correction consists of geometric transfer operators and direct coarse-grid discretization which is the analog of (4.2). The assumption ($A3$) is satisfied by averaging restriction for center residuals and injection for vertex residuals. By using Taylor expansion for the averaging restriction

$$\frac{1}{4}\begin{bmatrix} 1 & & 1 \\ & \bullet & \\ 1 & & 1 \end{bmatrix}, \tag{4.3}$$

for any function $\phi \in C^2(\Omega)$, we have

$$I_h^H I^h \phi = I^H \phi + \frac{h^2}{8} I^H \left[\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right] + o(h^2), \tag{4.4}$$

which leads to $r = 2$. The assumption ($A4$) for the fine-to-coarse solution transfer $\hat{I}_h^H$ to calculate the truncation error estimate is fulfilled by a fourth-order ($t = 4$) interpolation in one dimension: at the interior points this is given by

$$\frac{1}{16}\left[\begin{array}{ccccc} -1 & 9 & \bullet & 9 & -1 \end{array}\right], \tag{4.5}$$

and adjacent to the boundaries the stencil is modified to

$$\frac{1}{16}\left[\begin{array}{ccccc} 5 & \bullet & 15 & -5 & 1 \end{array}\right] \text{ or } \frac{1}{16}\left[\begin{array}{ccccc} 1 & -5 & 15 & \bullet & 5 \end{array}\right]. \tag{4.6}$$

The right hand sides of the system and the boundary data are chosen to match the analytical solutions $u(x, y) = e^{1.8u}\cos(3.7y)$ and $v(x, y) = -\cos(1.8x)\sin(3.7y)$. The discrete problem is solved by enough multigrid V-cycles that the solutions have effectively converged on the fine grid, thus satisfying assumption ($A2$). Consequently, according to (3.5), the τ-extrapolation (3.4) produces an approximation which is $O(h^4)$.

Figure. 4.2 shows results for the u equation, which verify the conclusion of the theorem; results for the v equation (not shown) are similar. In the figure, various measures of the error in the truncation error estimates, i.e., the left-hand side of (3.1) are plotted as functions of N. The truncation error is about $O(h^2)$ and the error in the truncation error estimates is about $O(h^4)$ except the maximum error over the whole domain which is about $O(h^3)$. This is due to the large error of the points near the boundary. The relative poor performance near the boundary does not invalidate the analysis in section 3: pointwise, the errors are $O(h^4)$ as expected.

5. Stokes equations. As a prelude to nonlinear Navier-Stokes equations, we first consider the linear Stokes equations,

$$-\Delta u + p_x = F_1, \tag{5.1a}$$

$$-\Delta v + p_y = F_2, \tag{5.1b}$$

$$u_x + v_y = F_3, \tag{5.1c}$$

on a domain $\Omega = [0, 1] \times [0, 1]$ with Dirichlet boundary conditions: specifying the true solution for u on east and west boundary, and the true solution of v on north and

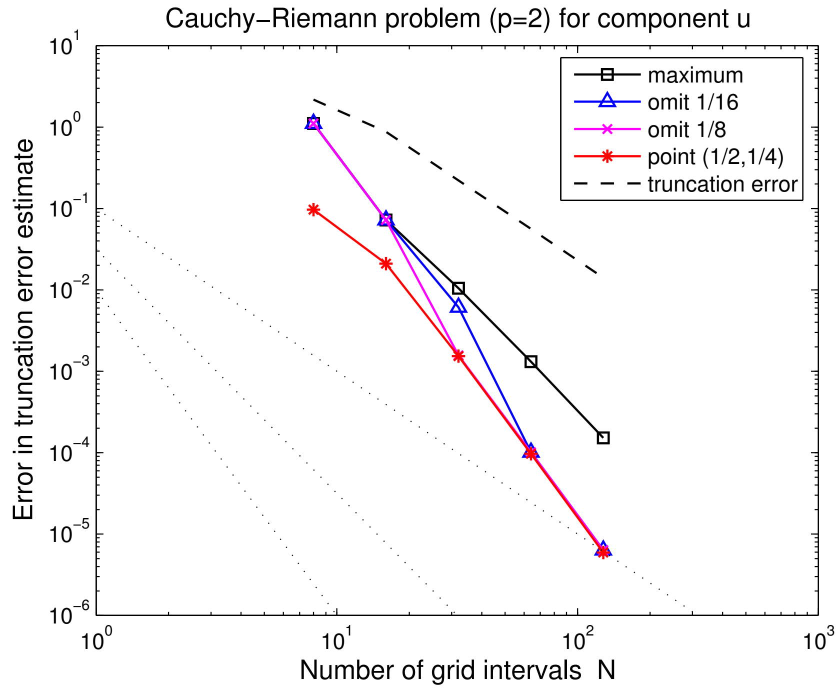

FIG. 4.2. *Error in the truncation error estimates (solid lines) and true truncation error (dashed lines) for the Cauchy-Riemann equation (4.1a). The curves give the maximum error over the domain, the maximum errors with strips of widths* $1/16$ *and* $1/8$ *adjacent to the boundaries omitted, and the errors at the point* $(1/2, 1/4)$. *Dotted lines give slopes for orders* $p = 2, 3$ *and* 4.

south boundary, where Δ is the Laplacian operator, (u, v) represents the velocity of a fluid, p represents the pressure, and F_i are given forcing functions. The discrete approximation for (5.1) on a staggered grid (Fig. 4.1) is,

$$-\Delta^h u^h + \partial_x^h p^h = F_1^h \qquad \text{at } u\text{-face centers,} \tag{5.2a}$$

$$-\Delta^h v^h + \partial_y^h p^h = F_2^h \qquad \text{at } v\text{-face centers,} \tag{5.2b}$$

$$\partial_x u^h + \partial_y v^h = F_3^h \qquad \text{at cell centers,} \tag{5.2c}$$

where Δ^h is the usual 5-point approximation for Laplace operator. For a point near a boundary, this approximation may involve a ghost point, which is obtained by a quadratic extrapolation from the boundary value and the first two nearby interior points. The other operators are the same as described in the Cauchy-Reimann equations. The order of the truncation errors in Stokes equations are also $O(h^2)$, and $(A1)$ holds with $p = 2$ and $q = 2$.

Since the Stokes equations are linear partial differential equations, we still use a linear multigrid correction scheme, in which the smoother is DGS. We use bilinear interpolation for u, v, p. The residual restriction for the $u-$ and $v-$ face centers is given by

$$I_{h,u}^H = \frac{1}{8}\begin{bmatrix} 1 & 2 & 1 \\ & \bullet & \\ 1 & 2 & 1 \end{bmatrix}, I_{h,v}^H = \frac{1}{8}\begin{bmatrix} 1 & & 1 \\ 2 & \bullet & 2 \\ 1 & & 1 \end{bmatrix}. \tag{5.3}$$

The residual restriction for cell centers is the same as (4.3) which is second order. Using the Taylor expansions we can show that for any function $\phi \in C^2(\Omega)$,

$$I_{h,u}^H I^h \phi = I^H \phi + \frac{h^2}{8} I^H \left[2\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right] + o(h^2). \tag{5.4}$$

Similarly, $I_{h,v}^H$ is also second-order restriction. Thus, $(A3)$ is satisfied with $r = 2$. For assumption $(A4)$ with higher-order t can be satisfied by a fourth-order full weighting restriction for u, v via the tensor product between (4.5-4.6) and a fourth-order interpolation the in other direction, i.e.,

$$\frac{1}{16} \left[\begin{array}{ccccc} -1 & 4 & 8 & 4 & -1 \end{array} \right], \tag{5.5}$$

and for p via the tensor product among (4.5–4.6).

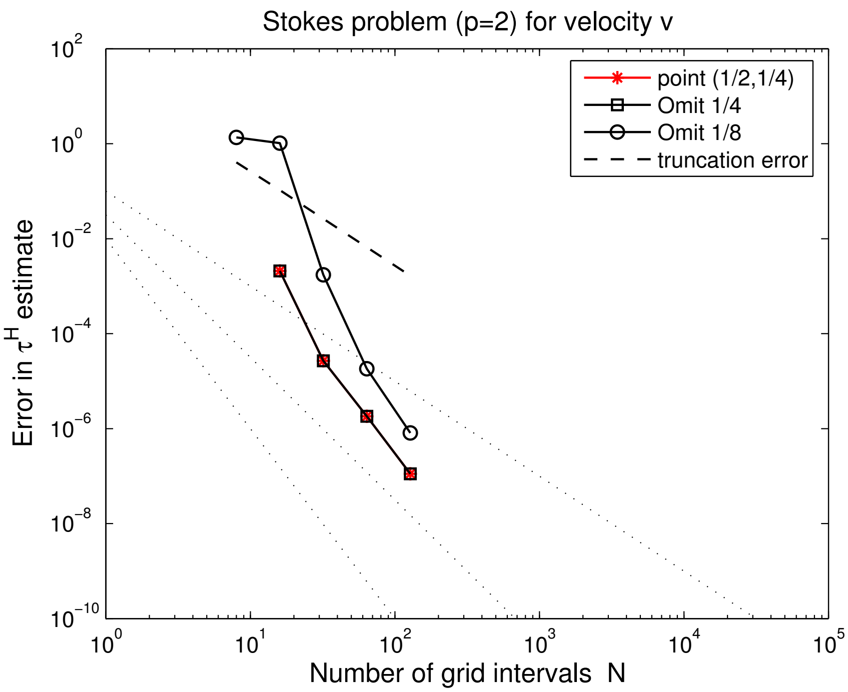

FIG. 5.1. *Error in the truncation error estimates (solid lines) and true truncation error (dashed lines) for the Stokes second momentum equation (5.2b). The curves give the maximum errors with strips of widths* $1/4$ *and* $1/8$ *adjacent to the boundaries omitted, and the errors at the point* $(1/2, 1/4)$.

The conclusion is illustrated by the numerical results in Fig. 5.1–5.2. Here, the forcing f and boundary data are chosen to match the analytical $u = \sin(3x + 3y)$, $v = -\sin(3x + 3y)$, and $p = 6\cos(3x + 3y)$ as used in [17]. Various measures of the error in the truncation error estimates, i.e., the left side of (3.1), are plotted as functions of N. Fig. 5.1 corresponds to the second momentum equation (the results corresponding to the first momentum equation are similar). The truncation error for the second momentum equation is $O(h^2)$, and the error in the truncation error estimates, if a narrow boundary strip (width 1/8 or 1/4) is omitted, is asymptotically

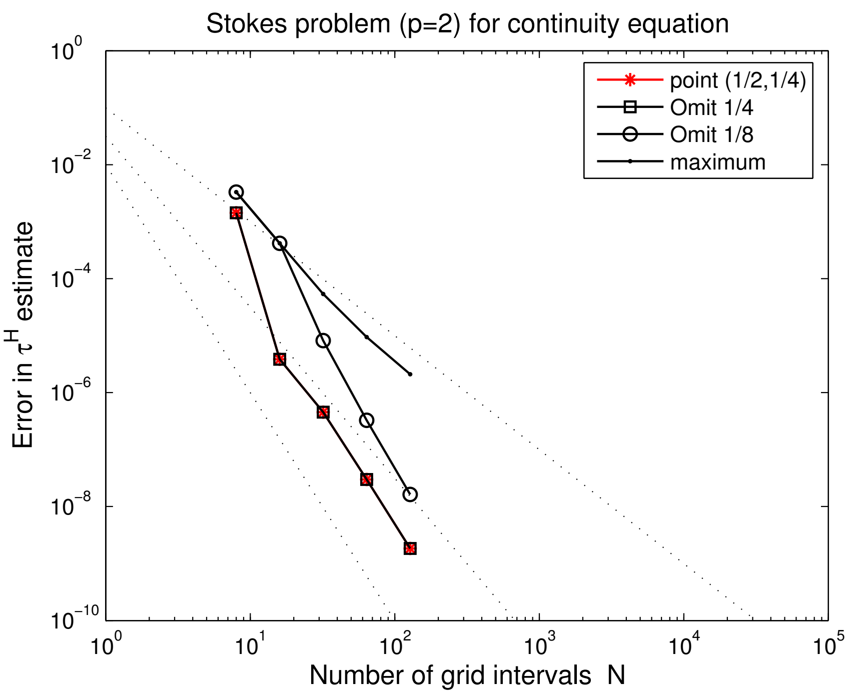

FIG. 5.2. *Error in the truncation error estimates for Stokes continuity equation (5.2c). Omit* $1/4$ *or* $1/8$ *is the maximum norm for a domain whose boundary strip (width* $1/4$ *or* $1/8$*) is omitted, and point* $(1/2, 1/4)$ *is the error evaluated at the point. Maximum is the maximum norm over the whole domain.*

$O(h^4)$—as it is at a typical point $(x, y) = (\frac{1}{2}, \frac{1}{4})$ as well. Fig. 5.2 shows the continuity equation. Since we chose the true solution which gives zero truncation error for the continuity equation, the truncation error is not included. However, the errors in the truncation error estimates also appear to be $O(h^4)$.

6. Navier-Stokes equations. An example of nonlinear system is given by Navier-Stokes equations

$$(u^2)_x + (uv)_y - \frac{1}{Re}\Delta u + p_x = F_1, \tag{6.1a}$$

$$(uv)_x + (v^2)_y - \frac{1}{Re}\Delta v + p_y = F_2, \tag{6.1b}$$

$$u_x + v_y = F_3, \tag{6.1c}$$

on a domain $\Omega = [0, 1] \times [0, 1]$ with Dirichlet boundary condition which is specified as for the Stokes equations, where Re is the Reynolds number. We discretize (6.1) on a staggered grid (Fig. 4.1) by using central differences for the nonlinear parts for a comparably slow flow, i.e.,

$$\partial_x^h(\mu_x^h u^h)^2 + \partial_y^h(\mu_y^h u^h \mu_x v^h) - \frac{1}{Re}\Delta^h u^h + \partial_x p^h = F_1 \qquad \text{at } u\text{-face centers} \tag{6.2a}$$

$$\partial_x^h(\mu_y^h u^h \mu_x v^h) + \partial_y^h(\mu_y^h v^h)^2 - \frac{1}{Re}\Delta^h v^h + \partial_y p^h = F_2 \qquad \text{at } v\text{-face centers} \tag{6.2b}$$

$$\partial_x^h u^h + \partial_y^h v^h = F_3, \tag{6.2c}$$

where $\mu_x^h = \frac{1}{2}(\phi(x+\frac{h}{2}, y) + \phi(x-\frac{h}{2}, y))$, and μ_y^h is defined similarly. After this discretization, the truncation error is $p = 2$. $(A3)$ is satisfied with $r = 2$ by a second-order residual transfer operator. Because of the nonlinearity, a FAS multigrid method is applied here. The smoothing operator is still DGS and when updating p^h for the continuity equation, we only consider the linear part with Laplace operator (which restricts big Reynolds number in this solver). The other multigrid components are the same as those used in Stokes system. The restriction for solution inside the FAS is the same as the restriction for the residual (4.3) and (5.3). Note that this does not have to be high order. For $\hat{I}_h^H$, we use the same fourth-order full weighting restriction as for the Stokes equations. Thus, $(A4)$ is satisfied by $t = 4$. In this numerical

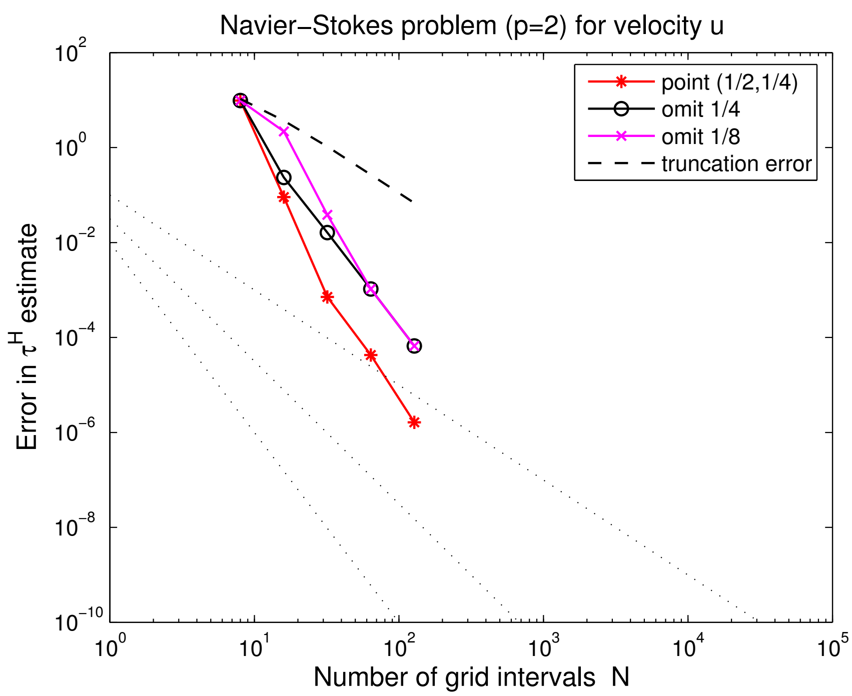

FIG. 6.1. *Error in the truncation error estimates (solid lines) and truncation error (dash line) for Navier-Stokes first momentum equation (6.2). Omit* 1/4 *or* 1/8 *is the maximum norm for a domain whose boundary strip (width* 1/4 *or* 1/8*) is omitted, and point* (1/2, 1/4) *is the error evaluated at the point.*

test, the Reynolds number is $Re = 1$ and the right hand side is chosen to fit for the exact solution $u = \sin(2\pi x)\cos(2\pi y)$, $v = -\cos(2\pi x)\sin(2\pi y)$, and $p = 0$. Similar results are obtained for other Reynolds number (e.g. $Re = 5, 10$) when the values are good for the central difference scheme and designed multigrid solver. Fig. 6.1

also demonstrates a high order $\alpha = 4$ is obtained in the error in the truncation error estimates.

7. Discussion and Conclusions. In [11], Fulton gave a proper definition of the relative truncation error on a non-staggered grid which eliminates the requirement of $I^H f = I_h^H I^h f$. However, [11] and other papers [1, 14, 16] also requires $\hat{I}^H u = \hat{I}_h^H \hat{I}^h u$ in the truncation error estimates, which leads to a dilemma: using injection on non-staggered grid [11] or using both high-order restriction and high-order discretization on coarse grid [1]. In our study, high-order truncation error estimates can be obtained by a high-order $\hat{I}_h^H$ without requiring the condition $\hat{I}^H u = \hat{I}_h^H \hat{I}^h u$. Thus, the truncation error estimates in [11] are extended to nonlinear problems on staggered grids with general multigrid components (simple discretizations on the coarse grid and second-order restrictions for both residue and solution). The only extra work we need to add is to use a high-order solution transfer in calculating the truncation error estimates. All three numerical examples verify the results.

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