

V-CYCLE MULTIGRID CONVERGENCE ANALYSIS: COMPARISON OF BOUNDS AND RELATION WITH THE TWO-GRID CONVERGENCE FACTOR

ARTEM NAPOV* AND YVAN NOTAY†

Abstract. We consider multigrid methods with V-cycle for symmetric positive definite linear systems. We consider bounds on the convergence factor that are characterized by a constant which is the maximum over all levels of an expression involving only two consecutive levels. More particularly, we consider the classical bound by Hackbusch, a bound by McCormick, and a bound obtained by applying the successive subspace correction convergence theory with so-called a -orthogonal decomposition. We show that the constants in these bounds are closely related, and hence that these analyses are equivalent from the qualitative point of view, whereas McCormick bound is in fact the best one. We also show some relation with the two-grid convergence factor that helps to understand when an optimal two-grid method leads to a multigrid method which is optimal with V-cycle. Moreover, it turns out that when Fourier analysis can be applied to estimate the two-grid convergence factor, one can derive with little additional effort a bound on the convergence factor for the multigrid method with V-cycle. Considering a typical example, we further show that this bound can provide a satisfactorily sharp estimate of the actual multigrid convergence speed.

Key words. multigrid, V-cycle, successive subspace correction, approximation property, convergence analysis

AMS subject classifications. 65F10, 65N55, 65F50

1. Introduction. We consider multigrid methods for solving symmetric positive definite (SPD) $n \times n$ linear systems

$$Ax = b. \tag{1.1}$$

Multigrid methods are based on the recursive use of a two-grid scheme. A basic two-grid method combines the action of a *smoother*, often a simple iterative method such as Gauss-Seidel, and a *coarse grid correction*, which involves solving a smaller problem on a coarser grid. A V-cycle multigrid method is obtained when this coarse problem is solved approximately with 1 iteration of the two-grid scheme on that level, and so on, until the coarsest level on which an exact solve is performed. Other cycles may be defined, for instance the W-cycle based on two stationary iterations at each level, see, e.g., [19].

If there are only two levels, accurate bounds may be obtained either by means of Fourier analysis [18, 19, 20], or using some appropriate algebraic tools [2, 3, 4, 13, 17]. This focus on two-grid schemes is motivated by the fact that, “if the two-grid method converges sufficiently well, then the multigrid method with W-cycle will have similar convergence properties” [19, p. 77] (see also [1, pp. 226–228] and [14]). This is not the case for the V-cycle since there are known examples where the two-grid method converges relatively well, whereas the multigrid method with V-cycle scales poorly with the number of levels [9]. Hence V-cycle analysis has to be, to some point, essentially different from two-grid analysis.

Multigrid methods with V-cycle have been analyzed in several ways. Recently, a sharp identity has been obtained [22], which may be seen as an improvement of the successive subspace correction

*Université Libre de Bruxelles, Service de Métrologie Nucléaire (C.P. 165-84), 50 Av. F.D. Roosevelt, B-1050 Brussels, Belgium. anapov@ulb.ac.be. Research supported by the Belgian FRNS (“Aspirant”).

†Université Libre de Bruxelles, Service de Métrologie Nucléaire (C.P. 165-84), 50 Av. F.D. Roosevelt, B-1050 Brussels, Belgium. ynotay@ulb.ac.be, homepages.ulb.ac.be/~ynotay. Research supported by the Belgian FRNS (“Directeur de recherches”).

(SSC) convergence theory (e.g., [5, 15, 16, 21, 23]). Whenever combined with the so-called a -orthogonal decomposition, this theory yields an estimate which is characterized by a constant that is the maximum over all levels of an expression involving only two consecutive levels. Optimal convergence can then be proved with an analysis which, like the standard two-grid analysis, involves only two-levels. This property is shared by Hackbusch classical bound for symmetric positive definite matrices [6, Theorem 7.2.2], and by a bound by McCormick [8].

Although these bounds are relatively old, there are few related questions so far left unanswered: how do they compare to each other? can they be related to the two-grid convergence factor? can they lead to a satisfactorily sharp estimate of the actual multigrid convergence? These questions are addressed in, respectively, Sections 3, 4 and 5 below, Section 2 being devoted to the reminder of previous results.

Note that the SSC theory is traditionally formulated in abstract setting, which does not facilitate the comparison with other theories. The results below are based on the algebraic formulation of the SSC theory as obtained in [10]. On the other hand, this short paper is an extended summary of two papers [11, 12], to which we refer for the proofs and some additional developments.

Notation. Let I denote the identity matrix and O the zero matrix. For any square matrix C , $\rho(C)$ is its spectral radius (that is, its largest eigenvalue in modulus); $\|C\| = \sqrt{\rho(C^T C)}$ is the usual 2-norm and, for a SPD matrix D , $\|C\|_D = \|D^{1/2} C D^{-1/2}\|$ is the D -norm (if $D = A$, it is also called energy norm).

2. Bounds on the V-cycle multigrid convergence factor.

2.1. General setting. We assume the system matrix A symmetric positive definite (SPD). We consider a multigrid method with $J+1$ levels ($J \geq 1$); index J refers to the finest level (on which the system (1.1) is to be solved), and index 0 to the coarsest level. The number of unknowns at level k , $0 \leq k \leq J$, is denoted by n_k (thus $n_J = n$). We consider a symmetric scheme based on the Galerkin principle; that is, the restriction is the transpose of the prolongation and the matrix A_k at level k , $k = J-1, \dots, 0$, is given by $A_k = P_k^T A_{k+1} P_k$, where P_k is the prolongation from level k to level $k+1$; we also assume that the smoother M_k is SPD and that one pre- and one post-smoothing step are performed on each level (see [11, 12] for more than one pre- and post-smoothing step).

The algorithm for V-cycle multigrid is then as follows.

Multigrid with V-cycle at level k : $x_{n+1} = \text{MG}(b, A_k, x_n, k)$

- (1) Relax with smoother M_k : $\bar{x}_n = \text{Smooth}(x_n, A_k, M_k, b)$
- (2) Compute residual: $r_k = b - A_k \bar{x}_n$
- (3) Restrict residual: $r_{k-1} = P_{k-1}^T r_k$
- (4) Coarse grid correction: **if** $k = 1$, $e_0 = A_0^{-1} r_0$
else $e_{k-1} = \text{MG}(r_{k-1}, A_{k-1}, 0, k-1)$
- (5) Prolongate coarse grid correction: $\hat{\hat{x}}_n = \bar{x}_n + P_{k-1} e_{k-1}$
- (6) Relax with smoother M_k : $x_{n+1} = \text{Smooth}(\hat{\hat{x}}_n, A_k, M_k, b)$

When applying this algorithm the error satisfies

$$A_k^{-1} b - x_{n+1} = E_{MG}^{(k)} (A_k^{-1} b - x_n)$$

where the iteration matrix $E_{MG}^{(k)}$ is recursively defined from

$$\begin{aligned} E_{MG}^{(0)} &= O \quad \text{and, for } k = 1, 2, \dots, J : \\ E_{MG}^{(k)} &= (I - M_k^{-1} A_k) \left(I - P_{k-1} (I - (E_{MG}^{(k-1)})) A_{k-1}^{-1} P_{k-1}^T A_k \right) (I - M_k^{-1} A_k) \end{aligned} \quad (2.1)$$

(see, e.g., [19, p. 48]). Our main objective is the analysis of the spectral radius of $E_{MG}^{(J)}$, which governs the convergence on the finest level. We make use of the following general assumptions.

General assumptions

- $n = n_J > n_{J-1} > \dots > n_0$;
- P_k is an $n_{k+1} \times n_k$ matrix of rank n_k , $k = J-1, \dots, 0$;
- $A_J = A$ and $A_k = P_k^T A_{k+1} P_k$, $k = J-1, \dots, 0$;
- M_k is SPD and such that $\forall w_k \in \mathbb{R}^{n_k} : w_k^T A_k w_k \leq w_k^T M_k w_k$, $k = J, \dots, 1$.

Note that SSC and McCormick theories allow for more general smoother scaling and require only $w_k^T A_k w_k \leq \omega w_k^T M_k w_k$ for some $\omega < 2$; we refer to [11, 12] for related developments.

We close this subsection by introducing the projector π_{A_k} which plays an important role throughout this paper:

$$\pi_{A_k} = P_{k-1} A_{k-1}^{-1} P_{k-1}^T A_k. \quad (2.2)$$

2.2. SSC theory. Theorem 2.1 below can be seen as an algebraic formulation (with some additional refinements) of the SSC theory as stated in [21, Theorem 4.4 and Lemma 4.6], and [23, Theorem 5.1]; we refer to [10] for a proof. Note that Theorem 2.1 contains some degrees of freedom, namely the matrices G_k , $k = J-1, \dots, 0$; in fact, the latter play the role of the space decomposition in the original (abstract) theory in [21, 23].

THEOREM 2.1. *Let $E_{MG}^{(J)}$ be defined by (2.1) with P_k , $k = 0, \dots, J-1$, A_k , $k = 0, \dots, J$, and M_k , $k = 1, \dots, J$, satisfying the general assumptions stated in Section 2.1, and set $M_0 = A_0$.*

Let G_k , $k = 0, \dots, J-1$, be $n_k \times n_{k+1}$ matrices, and, for $k = 0, \dots, J$, let \check{P}_k and \check{G}_k be defined by, respectively,

$$\begin{aligned} \check{P}_J &= I \\ \check{P}_k &= \check{P}_{k+1} P_k \quad , \quad k = J-1, \dots, 0, \end{aligned} \quad (2.3)$$

and

$$\begin{aligned} \check{G}_J &= I \\ \check{G}_k &= G_k \check{G}_{k+1} \quad , \quad k = J-1, \dots, 0, \end{aligned} \quad (2.4)$$

with $P_{-1} = G_{-1} = O$.

There holds

$$\rho(E_{MG}^{(J)}) \leq 1 - \frac{1}{K(1 + \|\Gamma\|)^2} \quad , \quad (2.5)$$

where

$$K = \max_{v \in \mathbf{R}^n} \frac{\sum_{k=0}^J v^T \check{G}_k^T (I - P_{k-1} G_{k-1})^T M_k (I - P_{k-1} G_{k-1}) \check{G}_k v}{v^T A v} \quad (2.6)$$

and

$$\Gamma = \begin{pmatrix} 0 & \gamma_{01} & \cdots & \gamma_{0J} \\ & 0 & \cdots & \gamma_{1J} \\ & & \ddots & \vdots \\ & & & 0 & \gamma_{(J-1)J} \\ & & & & 0 \end{pmatrix}, \quad (2.7)$$

with, for $k = 0, \dots, J-1$ and $l = k+1, \dots, J$,

$$\gamma_{kl} = \max_{w_k \in \mathbf{R}^{n_k}} \max_{v \in \mathbf{R}^n} \frac{v^T \check{G}_l^T (I - P_{l-1} G_{l-1})^T \check{P}_l^T A \check{P}_k w_k}{(w_k^T M_k w_k)^{1/2} (v^T \check{G}_l^T (I - P_{l-1} G_{l-1})^T M_l (I - P_{l-1} G_{l-1}) \check{G}_l v)^{1/2}}.$$

Moreover,

$$\|\Gamma\| \leq \sqrt{J(J+1)/2}. \quad (2.8)$$

Now, in this paper, we focus on bounds that can be estimated considering only two consecutive levels at a time. One can obtain such a bound from the above theorem if one can express $v^T A v$ as a sum similar to the one in the numerator of (2.6). In this view, consider

$$\begin{aligned} v^T A v &= v^T \left(\sum_{l=0}^J (\check{P}_l \check{G}_l - \check{P}_{l-1} \check{G}_{l-1})^T \right) A \left(\sum_{k=0}^J (\check{P}_k \check{G}_k - \check{P}_{k-1} \check{G}_{k-1}) \right) v \\ &= \sum_{k=0}^J v^T (\check{P}_k \check{G}_k - \check{P}_{k-1} \check{G}_{k-1})^T A (\check{P}_k \check{G}_k - \check{P}_{k-1} \check{G}_{k-1}) v \\ &\quad + 2 \sum_{k=0}^J \sum_{l=0}^{k-1} v^T (\check{P}_l \check{G}_l - \check{P}_{l-1} \check{G}_{l-1})^T A (\check{P}_k \check{G}_k - \check{P}_{k-1} \check{G}_{k-1}) v \\ &= \sum_{k=0}^J v^T \check{G}_k^T (I - P_{k-1} G_{k-1})^T A_k (I - P_{k-1} G_{k-1}) \check{G}_k v \\ &\quad + 2 \sum_{k=0}^J v^T (\check{P}_{k-1}^T \check{G}_{k-1})^T A (\check{P}_k \check{G}_k - \check{P}_{k-1} \check{G}_{k-1}) v \\ &= \sum_{k=0}^J v^T \check{G}_k^T (I - P_{k-1} G_{k-1})^T A_k (I - P_{k-1} G_{k-1}) \check{G}_k v \\ &\quad + 2 \sum_{k=0}^J v^T \check{G}_{k-1}^T P_{k-1}^T A_k (I - P_{k-1} G_{k-1}) \check{G}_k v. \end{aligned}$$

This expression has the desired form if the second term vanishes, which is obtained letting G_{k-1} be such that $P_{k-1}^T A_k P_{k-1} G_{k-1} = P_{k-1}^T A_k$; that is,

$$G_k = A_k^{-1} P_k^T A_{k+1} \quad (2.9)$$

and therefore $P_{k-1} G_{k-1} = \pi_{A_k}$. In fact, this choice of G_k corresponds to the so-called a -orthogonal decomposition in the original abstract theory. As shown in [10], it further follows that $\Gamma = 0$ in this case, and hence the following theorem holds.

THEOREM 2.2. Let $E_{MG}^{(J)}$ be defined by (2.1) with P_k , $k = 0, \dots, J-1$, A_k , $k = 0, \dots, J$, and M_k , $k = 1, \dots, J$, satisfying the general assumptions stated in Section 2.1.

Then,

$$\rho(E_{MG}^{(J)}) \leq 1 - \frac{1}{K}, \quad (2.10)$$

where

$$K = \max_{1 \leq k \leq J} \max_{v_k \in \mathbf{R}^{n_k}} \frac{v_k^T (I - \pi_{A_k})^T M_k (I - \pi_{A_k}) v_k}{v_k^T (I - \pi_{A_k})^T A_k (I - \pi_{A_k}) v_k} \quad (2.11)$$

with π_{A_k} defined by (2.2).

2.3. Hackbusch bound. The bound from [6, Theorem 7.2.2] is recalled in the following theorem.

THEOREM 2.3. Let $E_{MG}^{(J)}$ be defined by (2.1) with P_k , $k = 0, \dots, J-1$, A_k , $k = 0, \dots, J$, and M_k , $k = 1, \dots, J$, satisfying the general assumptions stated in Section 2.1.

Then,

$$\rho(E_{MG}^{(J)}) \leq \frac{c_A}{c_A + 2}, \quad (2.12)$$

where

$$c_A = \max_{1 \leq k \leq J} \max_{v_k \in \mathbf{R}^{n_k}} \frac{v_k^T (A_k^{-1} - P_{k-1} A_{k-1}^{-1} P_{k-1}^T) v_k}{v_k^T M_k^{-1} v_k}. \quad (2.13)$$

2.4. McCormick bound. The bound obtained in [8, Lemma 2.3, Theorem 3.4 and Section 5] (see also [7] for an alternative proof) is recalled in the following theorem.

THEOREM 2.4. Let $E_{MG}^{(J)}$ be defined by (2.1) with P_k , $k = 0, \dots, J-1$, A_k , $k = 0, \dots, J$, and M_k , $k = 1, \dots, J$, satisfying the general assumptions stated in Section 2.1.

Then,

$$\rho(E_{MG}^{(J)}) \leq 1 - \delta, \quad (2.14)$$

where

$$\delta = \min_{1 \leq k \leq J} \min_{v_k \in \mathbf{R}^{n_k}} \frac{\|v_k\|_{A_k}^2 - \|(I - M_k^{-1} A_k) v_k\|_{A_k}^2}{\|(I - \pi_{A_k}) v_k\|_{A_k}^2} \quad (2.15)$$

with π_{A_k} defined by (2.2).

3. Comparison. The following result relates the constants K , c_A and δ appearing in the different bounds. We refer to [11] for a proof.

THEOREM 3.1. Let K , c_A and δ be defined by, respectively, (2.11), (2.13) and (2.15), where P_k , $k = 0, \dots, J-1$, A_k , $k = 0, \dots, J$, and M_k , $k = 1, \dots, J$ satisfy the general assumptions stated in Section 2.1.

Then

$$K = c_A \quad (3.1)$$

and

$$\delta = \frac{1}{c_A^{(2)}}, \quad (3.2)$$

where

$$c_A^{(2)} = \max_{1 \leq k \leq J} \max_{v_k \in \mathbf{R}^{n_k}} \frac{v_k^T (A_k^{-1} - P_{k-1} A_{k-1}^{-1} P_{k-1}^T) v_k}{v_k^T \widetilde{M}_k^{-1} v_k} \quad (3.3)$$

with

$$\widetilde{M}_k = M_k (2M_k - A_k)^{-1} M_k. \quad (3.4)$$

Note that \widetilde{M}_k satisfies

$$I - \widetilde{M}_k^{-1} A_k = (I - M_k^{-1} A_k)^2 \quad (3.5)$$

and therefore $c_A^{(2)}$ can be viewed as a the approximation property constant (2.13) for *two* pre- and post-smoothing steps. In fact, as shown in [11] (based on the ideas expressed [8, Theorem 3.4]), there holds

$$\frac{c_A}{2} \leq c_A^{(2)} \leq \frac{1}{2} (c_A + 1). \quad (3.6)$$

This allows to prove the following corollary (see [11] for details), which provides a direct comparison of the three bounds (2.5), (2.12) and (2.14).

COROLLARY 3.2. *Let $E_{MG}^{(J)}$ be defined by (2.1) with P_k , $k = 0, \dots, J-1$, A_k , $k = 0, \dots, J$, and M_k , $k = 1, \dots, J$, satisfying the general assumptions stated in Section 2.1. Let K , c_A and δ be defined by, respectively, (2.11), (2.13) and (2.15).*

Then,

$$\rho(E_{MG}^{(J)}) \leq 1 - \delta \leq \min \left(1 - \frac{1}{K}, \frac{c_A}{c_A + 2} \right). \quad (3.7)$$

Moreover,

$$1 - \frac{1}{K} \leq 1 - \frac{1}{2\delta^{-1}} \quad (3.8)$$

and

$$\frac{c_A}{c_A + 2} \leq \frac{1}{1 + \delta}. \quad (3.9)$$

Hence one can see that McCormick bound is the best one but all three analyses are qualitatively equivalent and simultaneously succeed or fail to prove a significant (say, independent of the mesh size) convergence result.

4. Relation with the two-grid rate. As shown in the following theorem, the constant δ in McCormick bound can be related to the two-grid convergence factor; that is, the spectral radius of the iteration matrix

$$E_{TG}^{(k)} = (I - M_k^{-1}A_k) (I - P_{k-1}A_{k-1}^{-1}P_{k-1}^T A_k) (I - M_k^{-1}A_k), \quad k = 1, \dots, J \quad (4.1)$$

with just two consecutive levels (and exact solve on the coarse level). We refer to [12] for a proof.

THEOREM 4.1. *Let $E_{TG}^{(k)}$, $k = 1, \dots, J$ and δ be defined by, respectively, (4.1) and (2.15) with P_k , $k = 0, \dots, J-1$, A_k , $k = 0, \dots, J$, and M_k , $k = 1, \dots, J$ satisfying the general assumptions stated in Section 2.1.*

Then,

$$\delta^{-1} \leq \max_{1 \leq k \leq J} \frac{\|I - \pi_{A_k}\|_{\widetilde{M}_k}^2}{1 - \rho(E_{TG}^{(k)})} = \max_{1 \leq k \leq J} \frac{\|\pi_{A_k}\|_{\widetilde{M}_k}^2}{1 - \rho(E_{TG}^{(k)})}. \quad (4.2)$$

Moreover,

$$\delta^{-1} \geq \max_{1 \leq k \leq J} \max \left(\|\pi_{A_k}\|_{\widetilde{M}_k}^2, \frac{1}{1 - \rho(E_{TG}^{(k)})} \right). \quad (4.3)$$

Hence the theories considered in this paper prove a satisfactory convergence result for the V-cycle if and only if the two-grid method converges fast enough and $\|\pi_{A_k}\|_{\widetilde{M}_k} = \|\widetilde{M}_k^{1/2} \pi_{A_k} \widetilde{M}_k^{-1/2}\|$ is nicely bounded.

Often, a multigrid method is assessed by estimating the two-grid convergence rate with Fourier analysis [18, 19, 20]. It means that one considers a model constant coefficient PDE problem for which the eigenvectors of the discrete matrix are explicitly known at all levels. Simple smoothers have the same set of eigenvectors, and hence the matrices A_k and M_k are diagonal whenever expressed in the corresponding basis (the Fourier basis). Then, Fourier analysis is possible if and only if there exists an ordering and a partitioning of these eigenvectors for which P_{k-1} , expressed in the Fourier basis, has the form

$$P_{k-1} = \begin{pmatrix} p_1^{(k-1)} & & & \\ & p_2^{(k-1)} & & \\ & & \ddots & \\ & & & p_{n_{k-1}}^{(k-1)} \end{pmatrix},$$

where $p_i^{(k-1)}$ are vectors of size $m_l \times 1$ with small m_l (typically 4 for 2D problems). As is well known, it indeed follows that the two-grid iteration matrix (4.1) is then block diagonal with $m_l \times m_l$ diagonal blocks. Hence its spectral radius is the maximal eigenvalue in modulus of all these small blocks, which is easy to compute numerically, and which may sometimes be bounded analytically.

Now, observe that in this setting \widetilde{M}_k is also diagonal whenever expressed in the Fourier basis, and let

$$A_k = \begin{pmatrix} \Lambda_1^{(k)} & & & \\ & \Lambda_2^{(k)} & & \\ & & \ddots & \\ & & & \Lambda_{n_k}^{(k)} \end{pmatrix}, \quad \widetilde{M}_k = \begin{pmatrix} \Delta_1^{(k)} & & & \\ & \Delta_2^{(k)} & & \\ & & \ddots & \\ & & & \Delta_{n_k}^{(k)} \end{pmatrix}$$

be the related expression of A_k and \widetilde{M}_k , where the partitioning is consistent with that of P_{k-1} . Observing that $A_{k-1} = P_{k-1}^T A_k P_{k-1}$ is pointwise diagonal, one can see that $\widetilde{M}_k^{1/2} \pi_{A_k} \widetilde{M}_k^{-1/2}$ is also block diagonal with diagonal blocks of the form

$$\Delta_l^{(k) 1/2} p_l^{(k-1)} \left(p_l^{(k-1) T} \Lambda_l^{(k)} p_l^{(k-1)} \right)^{-1} p_l^{(k-1) T} \Lambda_l^{(k)} \Delta_l^{(k) -1/2}.$$

These are rank one matrices whose norm is easily expressed as (since $\|xy^T\| = \|x\| \|y\|$)

$$\frac{\| \Delta_l^{(k) 1/2} p_l^{(k-1)} \| \| p_l^{(k-1) T} \Lambda_l^{(k)} \Delta_l^{(k) -1/2} \|}{p_l^{(k-1) T} \Lambda_l^{(k)} p_l^{(k-1)}}.$$

To assess $\|\pi_{A_k}\|_{\widetilde{M}_k}$, one then needs only to estimate the norm of vectors of length m_l , which is in principle even easier than estimating the spectral radius of $m_l \times m_l$ matrices. Hence it should be possible in many cases to supplement the analysis of $\rho(E_{TG}^{(k)})$ with an analysis of $\|\pi_{A_k}\|_{\widetilde{M}_k}$, yielding a bound on the convergence factor of the multigrid method with V-cycle via Theorems 4.1 and 2.4.

5. Example.

5.1. Laplacian with standard geometric coarsening. We consider the linear system resulting from the bilinear finite element discretization of the two-dimensional Poisson problem

$$\begin{aligned} -\Delta u &= f & \text{in } \Omega &= (0, 1) \times (0, 1) \\ u &= 0 & \text{in } \partial\Omega \end{aligned}$$

on a uniform grid of mesh size $h = 1/N_J$ in both directions. The matrix corresponds then to the following nine point stencil

$$\begin{bmatrix} -1 & -1 & -1 \\ -1 & 8 & -1 \\ -1 & -1 & -1 \end{bmatrix}. \quad (5.1)$$

Up to some scaling factor, this is also the stencil obtained with 9-point finite difference discretization.

We assume $N_J = 2^J N_0$ for some integer N_0 , allowing J steps of regular geometric coarsening. We consider the standard prolongation

$$P_k = \begin{pmatrix} J_k \\ I_{n_k} \end{pmatrix},$$

where J_k corresponds to the natural interpolation associated with bilinear finite element basis functions. The restriction P_k^T corresponds then to ‘‘full weighting’’, as defined in, e.g. [19]¹. It then follows that all successive coarse grid matrices (computed with Galerkin formula) also correspond to the stencil (5.1).

We consider damped Jacobi smoothing: $M_k = \omega_{\text{Jac}}^{-1} \text{diag}(A_k)$. Since $\sigma(A_k) \in (0, 12)$, the smoother M_k satisfy the general assumptions stated in Section 2.1 if $\omega_{\text{Jac}}^{-1} > \frac{3}{2}$.

¹up to some scaling factor; the scalings of the prolongation and restriction are unimportant when using coarse grid matrices of the Galerkin type.

Using Fourier analysis, one can show that [12], for $J \rightarrow \infty$,

$$\begin{aligned} K = c_A &= \frac{4}{3\omega_{\text{Jac}}} \\ \delta^{-1} &= \frac{1}{3\omega_{\text{Jac}}} + \frac{1}{1 - (1 - 3\omega_{\text{Jac}}/2)^2} \\ \|\pi_A\|_{\tilde{M}}^2 &= \max_{1 \leq k \leq J} \|\pi_{A_k}\|_{\tilde{M}_k}^2 = 2 - \frac{3}{4\omega_{\text{Jac}}}. \end{aligned}$$

Combined with the measured asymptotic two-grid rate $\rho(E_{TG}) = \max_{1 \leq k \leq J} \rho(E_{TG}^{(k)})$, this leads to the bounds reported in Table 5.1, where they are compared with the actual convergence rate. One sees that McCormick bound is indeed the best one. Moreover, either directly, or indirectly via Theorem 4.1, it provides a satisfactorily sharp prediction of the actual multigrid convergence rate.

ω_{Jac}	$1 - \frac{1}{K}$	$\frac{c_A}{c_A+2}$	$1 - \delta$	$\rho(E_{TG})$	$1 - \frac{1 - \rho(E_{TG})}{\ \pi_A\ _{\tilde{M}}^2}$	$\rho(E_{MG}^{(J)})$
0.500	0.626	0.571	0.423	0.391	0.625	0.398
0.667	0.5	0.5	0.333	0.250	0.5	0.271

TABLE 5.1

Convergence factor for V -cycle multigrid and related bounds; $\rho(E_{MG}^{(J)})$ has been computed for $N_0 = 2$ and $J = 6$, whereas other values are asymptotic ($J \rightarrow \infty$).

5.2. Laplacian with aggregation-based coarsening. We consider the linear system resulting from the linear finite element discretization of the one-dimensional Poisson problem with periodic boundary conditions

$$\begin{aligned} -\frac{d^2 u}{dx^2} &= f \quad \text{in } \Omega = (0, 1) \\ u(1) &= u(0) \end{aligned}$$

on a uniform grid of mesh size $h = 1/(N - 1)$ with $N = 2^J N_0$. This leads to the following stencil

$$\begin{bmatrix} -1 & 2 & -1 \end{bmatrix}. \quad (5.2)$$

We consider the piece-wise constant prolongation

$$P_k = \begin{pmatrix} 1 & 1 & & & \\ & & \ddots & & \\ & & & & 1 & 1 \end{pmatrix}^T. \quad (5.3)$$

All successive coarse grid matrices correspond then to the stencil (5.2), up to some scaling factor [12]. We consider damped Jacobi smoothing: $M_k = 2\text{diag}(A_k)$.

Using Fourier analysis, it can then be shown that [12]

$$\begin{aligned} \|\pi_A\|_{\tilde{M}}^2 &= \max_{1 \leq k \leq J} \|\pi_{A_k}\|_{\tilde{M}_k}^2 = \mathcal{O}(h^{-2}) \\ \rho(E_{TG}) &= \max_{1 \leq k \leq J} \rho(E_{TG}^{(k)}) = \frac{1}{2} + \mathcal{O}(h^2). \end{aligned}$$

Hence the two-grid method is optimal, but $\|\pi_A\|_{\widetilde{M}}$ is unbounded, which may indicate a bad behavior of the multigrid method with V-cycle. This is, indeed, what happens, as one can see with the numerical results reported in Table 5.2.

J	0	2	4	6	8
$\rho(E_{MG}^{(J)})$	0.375	0.800	0.947	0.986	0.997

TABLE 5.2

Convergence factor for V-cycle multigrid with $N_0 = 4$ and increasing J .

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