AN APPROXIMATE INVERSE-BASED PRECONDITIONER FOR INCOMPRESSIBLE MAGNETOHYDRODYNAMICS

MICHAEL WATHEN* AND CHEN GREIF†

Abstract. We consider an approximate inverse preconditioner for a mixed finite element discretization of an incompressible magnetohydrodynamics (MHD) problem. The derivation relies on the nullity of the discrete curl-curl operator in the Maxwell subproblem. We obtain a formula for the inverse that contains zero blocks, and use discretization considerations to sparsify the formula to develop a practical preconditioner. We demonstrate the viability of our approach with a set of preliminary numerical experiments.

Key words. incompressible magnetohydrodynamics, saddle-point linear systems, null space, preconditioners, approximate inverse, Krylov subspace methods

AMS Subject Classification. 65F08, 65F10, 65F50, 65N22

1. Introduction.

Given a sufficiently smooth domain Ω, consider the steady-state incompressible magnetohydrodynamics (MHD) model [1, Ch. 2]:

$$-\nu\,\Delta \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p - \kappa\,(\nabla\times\boldsymbol{b})\times\boldsymbol{b} = \boldsymbol{f} \qquad \text{in } \Omega, \tag{1.1a}$$

$$\nabla\cdot\boldsymbol{u} = 0 \qquad \text{in } \Omega, \tag{1.1b}$$

$$\kappa\nu_m\,\nabla\times(\nabla\times\boldsymbol{b}) + \nabla r - \kappa\,\nabla\times(\boldsymbol{u}\times\boldsymbol{b}) = \boldsymbol{g} \qquad \text{in } \Omega, \tag{1.1c}$$

$$\nabla\cdot\boldsymbol{b} = 0 \qquad \text{in } \Omega. \tag{1.1d}$$

Here $\boldsymbol{u}$ is the velocity, p the hydrodynamic pressure, $\boldsymbol{b}$ is a magnetic field, and the Lagrange multiplier associated with the divergence constraint on the magnetic field is denoted by r. The functions $\boldsymbol{f}$ and $\boldsymbol{g}$ represent external forcing terms.

To complete the model, we consider the following homogeneous Dirichlet boundary conditions:

$$\boldsymbol{u} = \boldsymbol{0} \qquad \text{on } \partial\Omega, \tag{1.2a}$$

$$\boldsymbol{n}\times\boldsymbol{b} = \boldsymbol{0} \qquad \text{on } \partial\Omega, \tag{1.2b}$$

$$r = 0 \qquad \text{on } \partial\Omega, \tag{1.2c}$$

with $\boldsymbol{n}$ being the unit outward normal on $\partial\Omega$.

We will consider a finite element discretization of the MHD model (1.1)–(1.2). Let us denote the L^2-inner product on $L^2(\Omega)^d$ by $(\cdot,\cdot)_\Omega$, for $d = 2, 3$. We introduce the standard Sobolev spaces as:

$$\begin{aligned}
\boldsymbol{V} &= H_0^1(\Omega)^d = \left\{\boldsymbol{u}\in H^1(\Omega)^d \,:\, \boldsymbol{u} = \boldsymbol{0} \text{ on } \partial\Omega\right\},\\
Q &= L_0^2(\Omega) = \{p \in L^2(\Omega) \,:\, (p\,,1)_\Omega = 0\},\\
\boldsymbol{C} &= H_0(\mathrm{curl};\Omega) = \left\{\boldsymbol{b}\in L^2(\Omega)^d \,:\, \nabla\times\boldsymbol{b}\in L^2(\Omega)^{\bar{d}},\ \boldsymbol{n}\times\boldsymbol{b} = \boldsymbol{0} \text{ on } \partial\Omega\right\},\\
S &= H_0^1(\Omega) = \{r\in H^1(\Omega) \,:\, r = 0 \text{ on } \partial\Omega\},
\end{aligned} \tag{1.3}$$

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where $\bar{d} = 2d - 3$ for 2D and 3D, which are the cases of interest. Using the weak formulation in [11] of the incompressible MHD system (1.1)–(1.2), the solution entails finding $(\boldsymbol{u}, p, \boldsymbol{b}, r) \in \boldsymbol{V} \times Q \times \boldsymbol{C} \times S$ such that

$$A(\boldsymbol{u}, \boldsymbol{v}) + O(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v}) + C(\boldsymbol{b}; \boldsymbol{v}, \boldsymbol{b}) + B(\boldsymbol{v}, p) = (\boldsymbol{f}, \boldsymbol{v})_{\Omega}, \tag{1.4a}$$

$$B(\boldsymbol{u}, q) = 0, \tag{1.4b}$$

$$M(\boldsymbol{b}, \boldsymbol{c}) - C(\boldsymbol{b}; \boldsymbol{u}, \boldsymbol{c}) + D(\boldsymbol{c}, r) = (\boldsymbol{g}, \boldsymbol{c})_{\Omega}, \tag{1.4c}$$

$$D(\boldsymbol{b}, s) = 0, \tag{1.4d}$$

for all $(\boldsymbol{v}, q, \boldsymbol{c}, s) \in \boldsymbol{V} \times Q \times \boldsymbol{C} \times S$. The variational forms are given by

$$\begin{aligned} A(\boldsymbol{u}, \boldsymbol{v}) &= \int_{\Omega} \nu \, \nabla \boldsymbol{u} : \nabla \boldsymbol{v} \, d\boldsymbol{x}, & B(\boldsymbol{u}, q) &= -\int_{\Omega} (\nabla \cdot \boldsymbol{u}) \, q \, d\boldsymbol{x}, \\ M(\boldsymbol{b}, \boldsymbol{c}) &= \int_{\Omega} \kappa \nu_m (\nabla \times \boldsymbol{b}) \cdot (\nabla \times \boldsymbol{c}) \, d\boldsymbol{x}, & D(\boldsymbol{b}, s) &= \int_{\Omega} \boldsymbol{b} \cdot \nabla s \, d\boldsymbol{x}, \\ O(\boldsymbol{w}; \boldsymbol{u}, \boldsymbol{v}) &= \int_{\Omega} (\boldsymbol{w} \cdot \nabla) \boldsymbol{u} \cdot \boldsymbol{v} \, d\boldsymbol{x}, & C(\boldsymbol{d}; \boldsymbol{v}, \boldsymbol{b}) &= \int_{\Omega} \kappa \, (\boldsymbol{v} \times \boldsymbol{d}) \cdot (\nabla \times \boldsymbol{b}) \, d\boldsymbol{x}. \end{aligned} \tag{1.5}$$

In a standard FEM fashion, we linearize around the current velocity and magnetic fields and introduce basis functions corresponding to the discrete spaces $\boldsymbol{V}_h, Q_h, \boldsymbol{C}_h$ and S_h of (1.3). This yields the following matrix system:

$$\begin{pmatrix} F(u) & B^T & C(b)^T & 0 \\ B & 0 & 0 & 0 \\ -C(b) & 0 & M & D^T \\ 0 & 0 & D & 0 \end{pmatrix} \begin{pmatrix} \delta u \\ \delta p \\ \delta b \\ \delta r \end{pmatrix} = \begin{pmatrix} r_u \\ r_p \\ r_b \\ r_r \end{pmatrix}, \tag{1.6}$$

with

$$\begin{aligned} r_u &= f - F(u)u - C(b)^T b - B^T p, \\ r_p &= -Bu, \\ r_b &= g - Mu + C(b)b - D^T r, \\ r_r &= -Db, \end{aligned}$$

where $F(u) = A + O(u)$. The matrices are: F, the discrete convection-diffusion operator; B, a discrete divergence operator; M, the discrete curl-curl operator; D, a discrete divergence operator; and C, a discrete coupling term. We define n_u, m_u, n_b and m_b as the dimension of the velocity, pressure, magnetic and multiplier variables, respectively.

In this paper we introduce an indefinite block preconditioner based on an approximate inverse for the system (1.6). We first derive a new formula for the inverse and show that the (exact) inverse has in fact a few zero blocks. We then approximate Schur complements that appear in the formula by sparse operators, and derive a new inverse formula. Numerical experiments demonstrate the viability and effectiveness of this preconditioning approach.

2. A new approximate inverse-based preconditioner.

Let us denote by $\mathcal{K}$ the coefficient matrix in the MHD model (1.6) and write it as:

$$\mathcal{K} = \begin{pmatrix} \mathcal{K}_{\text{NS}} & \mathcal{K}_{\text{C}}^T \\ -\mathcal{K}_{\text{C}} & \mathcal{K}_{\text{M}} \end{pmatrix},$$

where $\mathcal{K}_{\rm NS}$ is the Navier-Stokes subproblem, $\mathcal{K}_{\rm C}$ is the block for the coupling and $\mathcal{K}_{\rm M}$ is the Maxwell subproblem:

$$\mathcal{K}_{\rm NS} = \begin{pmatrix} F & B^T \\ B & 0 \end{pmatrix}, \quad \mathcal{K}_{\rm M} = \begin{pmatrix} M & D^T \\ D & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{K}_{\rm C} = \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix}.$$

Then, by [2, Equation (3.4)], the inverse is given by

$$\mathcal{K}^{-1} = \begin{pmatrix} \mathcal{K}_{\rm NS}^{-1} + \mathcal{K}_{\rm NS}^{-1}\mathcal{K}_{\rm C}^T\mathcal{S}^{-1}\mathcal{K}_{\rm C}{\mathcal{K}_{\rm NS}}^{-1} & -\mathcal{K}_{\rm NS}^{-1}\mathcal{K}_{\rm C}^T\mathcal{S}^{-1} \\ -\mathcal{S}^{-1}\mathcal{K}_{\rm C}{\mathcal{K}_{\rm NS}}^{-1} & \mathcal{S}^{-1} \end{pmatrix}, \tag{2.1}$$

where $\mathcal{S}$ denotes the Schur complement,

$$\mathcal{S} \;=\; \mathcal{K}_{\rm M} + \mathcal{K}_{\rm C}\mathcal{K}_{\rm NS}^{-1}\mathcal{K}_{\rm C}^T. \tag{2.2}$$

The inverses $\mathcal{K}_{\rm NS}^{-1}$ and $\mathcal{S}^{-1}$ appear multiple times in (2.1), and we now derive explicit formulas that further reveal their block structure. Notably, using results that have appeared in [4], we show that $\mathcal{S}^{-1}$ has a zero (2,2) block, and can be expressed in terms of a free matrix parameter. Let us write

$$\mathcal{K}_{\rm NS}^{-1} = \begin{pmatrix} K_1 & K_2 \\ K_3 & K_4 \end{pmatrix}. \tag{2.3}$$

We then have the following useful result.

Theorem 2.1. *Let $\mathcal{K}_{\rm NS}$ be written in block form as in* (2.3). *Then*

$$\mathcal{S}^{-1} = \begin{pmatrix} M_F^{-1}(I - D^TW^{-1}G^T) & GW^{-1} \\ W^{-1}G^T & 0 \end{pmatrix}, \tag{2.4}$$

where W is a (free) symmetric positive definite matrix,

$$M_F = M + D^TW^{-1}D + CK_1^{-1}C^T \quad \text{and} \quad G = M_F^{-1}D^T.$$

Proof. Writing out all the matrices involved in formula (2.2) for $\mathcal{S}$, we have

$$\mathcal{S} = \begin{pmatrix} M + CK_1C^T & D^T \\ D & 0 \end{pmatrix}. \tag{2.5}$$

Thanks to the curl appearance in the definition for C^T and M, in continuous form (1.1) and variational form, the null spaces of C^T and M are identical and are made up of discrete gradients. Therefore,

$$\dim(\text{null}(M + CK_1C^T)) = m_b,$$

where m_b was previously defined as the number of rows of the magnetic discrete divergence matrix D, and is equal to the dimension of the null space of the discrete curl operator, M. We thus have the exact same structure as in [4, equation (3.6)], and therefore the inverse of the Schur complement is given by (2.4). ∎

Using the inverse formula (2.1) and (2.4) together gives the exact expression for the inverse of (1.6) as

$$\mathcal{K}^{-1} = \begin{pmatrix} K_1 - K_1\hat{Z}K_1 & K_2 - K_1\hat{Z}K_2 & -K_1C^TM_F^{-1}H & 0 \\ K_3 - K_3\hat{Z}K_1 & K_4 - K_3\hat{Z}K_2 & -K_3C^TM_F^{-1}H & 0 \\ M_F^{-1}CK_1 & M_F^{-1}CK_2 & M_F^{-1}H & GW^{-1} \\ 0 & 0 & W^{-1}G^T & 0 \end{pmatrix}. \tag{2.6}$$

where $\hat{Z} = C^TM_F^{-1}C$ and $H = I - D^TW^{-1}G^T$.

The sparsity pattern of $\mathcal{K}^{-1}$, as expressed in (2.6), is illustrated in Figure 2.1.

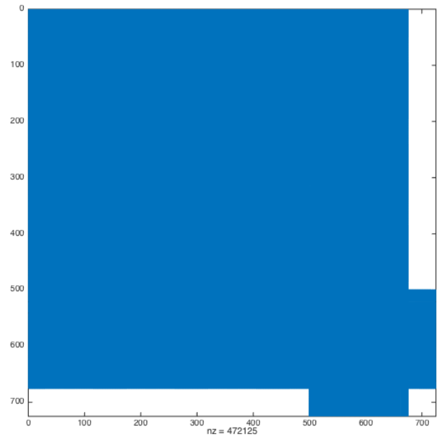


Fig. 2.1: Sparsity pattern of $\mathcal{K}^{-1}$.

2.1. A sparse approximation of the Schur complement. The presence of $CK_1^{-1}C^T$ within M_F is one of the bottlenecks in using the Schur complement, $\mathcal{S}$. Defining

$$M_F = M_W + CK_1^{-1}C^T$$

where $M_W = M + D^T W^{-1} D$, then using the Sherman-Morrison-Woodbury formula we can re-write M_F as

$$M_F^{-1} = M_W^{-1} - M_W^{-1}C(K_1 - C^T M_W^{-1} C)^{-1} C^T M_W^{-1}. \tag{2.7}$$

Using (2.7) for G in (2.4) we obtain

$$\begin{aligned} G &= (M_W^{-1} - M_W^{-1}C(K_1 - C^T M_W^{-1} C)^{-1} C^T M_W^{-1}) D^T, \\ &= M_W^{-1} D^T - M_W^{-1}C(K_1 - C^T M_W^{-1} C)^{-1} C^T M_W^{-1} D^T = M_W^{-1} D^T, \end{aligned}$$

since $M_W^{-1} D^T$ (being a definition of the discrete gradients from [4, Proposition 3.6]) is the null space matrix of M and C^T.

In order to approximate M_F^{-1} we again use the Sherman-Morrison-Woodbury formula (2.7). Using this formula we see that

$$M_W^{-1}C(K_1 - C^T M_W^{-1} C)^{-1} C^T M_W^{-1} \approx \mathcal{O}(h^4) \quad \text{and} \quad M_W \approx \mathcal{O}(h^2).$$

As we increase our problem/system size (small h), the dominant term is the $\mathcal{O}(h^2)$ M_W^{-1} term whereas the $\mathcal{O}(h^4)$ term acts as a small correction. Therefore, we take

$$M_F^{-1} \approx M_W^{-1},$$

where for small h the approximation gets better.

From [4] we take $DG = L$, recalling that G is the matrix of null vectors of M. Since G is made up of discrete gradients then from [6, Proposition 2.2], L is defined to be the scalar Laplacian on S_h. Also, shown in [6] the vector mass matrix, X on

$\boldsymbol{C}_h$, is spectrally equivalent to $D^T L^{-1} D$. Using these two results and the observation that a multiplication of either the inverse or block triangular preconditioner involves multiplications of the leading block with M ($G^T M = 0$), then the simplified inverse Schur complement becomes:

$$\mathcal{S}^{-1} \approx \mathcal{S}_{\text{approx}}^{-1} = \begin{pmatrix} M_X^{-1} & GL^{-1} \\ L^{-1}G^T & 0 \end{pmatrix}, \tag{2.8}$$

where $M_X = M + X$ and $G = M_X^{-1} D^T$.

2.2. A practical preconditioner. In a similar fashion to (2.8), we can reduce H to the identity (due to the multiplication of $G^T M = 0$). Also, we note that

$$K_i \hat{Z} K_j \gtrsim \mathcal{O}(h^3)$$

for any $i, j = 1, 2, 3, 4$. Hence, removing these terms we form the first step for the approximation of (2.6) as:

$$\hat{\mathcal{K}}^{-1} = \begin{pmatrix} K_1 & K_2 & -K_1 C^T M_X^{-1} & 0 \\ K_3 & K_4 & -K_3 C^T M_X^{-1} & 0 \\ M_X^{-1} C K_1 & M_X^{-1} C K_2 & M_X^{-1} & GL^{-1} \\ 0 & 0 & L^{-1} G^T & 0 \end{pmatrix}. \tag{2.9}$$

The final step to approximate (2.9), is to consider the inverse of the Navier-Stokes system $\mathcal{K}_{\text{NS}}$. For this we return to the exact inverse formula of a block matrix in (2.1). Applying this to the Navier-Stokes system gives the precise expression for the inverse

$$\mathcal{K}_{\text{NS}}^{-1} = \begin{pmatrix} F^{-1} - F^{-1} B^T S_{\text{NS}}^{-1} B F^{-1} & F^{-1} B^T S_{\text{NS}}^{-1} \\ S_{\text{NS}}^{-1} B F^{-1} & -S_{\text{NS}}^{-1} \end{pmatrix}. \tag{2.10}$$

Practically, we use the PCD approximation for S_{NS} given in (3.12) below. Substituting (2.10) into the expression for $\hat{\mathcal{K}}^{-1}$ in (2.9) gives

$$\hat{\mathcal{K}}^{-1} = \begin{pmatrix} N & F^{-1} B^T S_{\text{NS}}^{-1} & -N C^T M_X^{-1} & 0 \\ S_{\text{NS}}^{-1} B F^{-1} & -S_{\text{NS}}^{-1} & -S_{\text{NS}}^{-1} B F^{-1} C^T M_X^{-1} & 0 \\ M_X^{-1} C N & M_X^{-1} C F^{-1} B^T S_{\text{NS}}^{-1} & M_X^{-1} & GL^{-1} \\ 0 & 0 & L^{-1} G^T & 0 \end{pmatrix}, \tag{2.11}$$

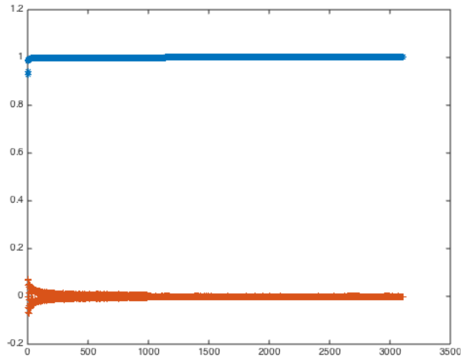
where

$$N = F^{-1} - F^{-1} B^T S_{\text{NS}}^{-1} B F^{-1}.$$

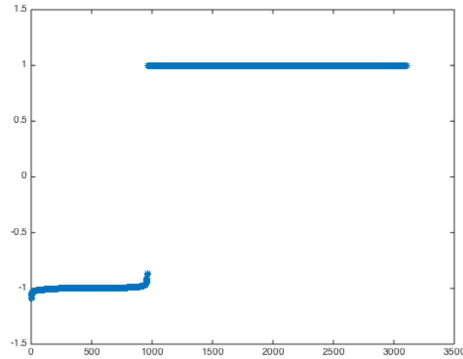
As with the approximation of M_F^{-1} in Section 2.1, we consider the approximate orders of the individual blocks of (2.11). Removing the $\mathcal{O}(h^3)$ terms in the (1,3) and (3,1) blocks of (2.11) yields the approximation:

$$\mathcal{P}_1^{-1} = \begin{pmatrix} F^{-1} - N & F^{-1} B^T S_{\text{NS}}^{-1} & 0 & 0 \\ S_{\text{NS}}^{-1} B F^{-1} & -S_{\text{NS}}^{-1} & -S_{\text{NS}}^{-1} B F^{-1} C^T M_X^{-1} & 0 \\ 0 & M_X^{-1} C F^{-1} B^T S_{\text{NS}}^{-1} & M_X^{-1} & GL^{-1} \\ 0 & 0 & L^{-1} G^T & 0 \end{pmatrix}. \tag{2.12}$$

Using $\mathcal{P}_1^{-1}$ as the preconditioner, we obtain the eigenvalue plot given in Figure 2.2(a). From the figure, we note that the red curve (the imaginary parts) are close to zero and hence we have very strong clustering of eigenvalues around one.



(a) Real (blue) and imaginary (red) part of eigenvalues of preconditioned matrix $\mathcal{P}_1^{-1}\mathcal{K}$.



(b) Eigenvalues of preconditioned matrix $\mathcal{P}_2^{-1}\mathcal{K}$.

Fig. 2.2: Preconditioned eigenvalue plots for approximate inverse (a) and block triangular (b) preconditioners

3. A block triangular preconditioner. As well as the approximate inverse preconditioner, we introduce a class of Schur complement based preconditioners for (1.6). We follow the well known setting of [8, 10] for experimental comparison. Let us define $\mathcal{P}_{\text{Block}}$ as:

$$\mathcal{P}_{\text{Block}} = \begin{pmatrix} \mathcal{K}_{\text{NS}} & \mathcal{K}_{\text{C}} \\ 0 & -\mathcal{S} \end{pmatrix}. \tag{3.1}$$

From [8, 10] the preconditioned matrix, $\mathcal{P}_{\text{Block}}^{-1}\mathcal{K}$, has precisely two eigenvalues ± 1 and is diagonalizable. We would therefore expect an appropriate Krylov subspace solver to converge within two iterations in exact precision. To use $\mathcal{P}_{\text{Block}}$ as a preconditioner, we require a direct Navier-Stokes solve and a Schur complement solve.

The direct solve for the Navier-Stokes system is too costly, so we approximate $\mathcal{K}_{\text{NS}}$ with the Schur complement system:

$$\mathcal{P}_{\text{NS}} = \begin{pmatrix} F & B^T \\ 0 & -S_{\text{NS}} \end{pmatrix}, \tag{3.2}$$

where $S_{\text{NS}} = BF^{-1}B^T$ is the fluid Schur complement. Using (3.2), obtains the more practical preconditioner

$$\hat{\mathcal{P}}_{\text{Block}} = \begin{pmatrix} \mathcal{P}_{\text{NS}} & \mathcal{K}_{\text{C}} \\ 0 & -\mathcal{S} \end{pmatrix}. \tag{3.3}$$

THEOREM 3.1. *The matrix $\hat{\mathcal{P}}_{\text{Block}}^{-1}\mathcal{K}$ has an eigenvalue $\lambda = 1$ of algebraic multiplicity at least n_u, and an eigenvalue $\lambda = -1$ of algebraic multiplicity at least n_b. The corresponding (known) eigenvectors are given as follows:*
$\lambda = 1$: with eigenvectors $\{v_i\}_{i=1}^{n_b-m_b}$ and $\{v_j\}_{j=n_b-m_b+1}^{n_u}$, as follows:

$$v_i = (u_i, -S^{-1}Bu_i, b_i, 0) \quad \textit{and} \quad v_j = (u_j, -S^{-1}Bu_j, 0, 0),$$

$$\textit{where } b_i \in \textit{null}(D) \neq 0,\ Cu_i = (2M + CK_1C^T)b_i \textit{ and } u_j \in \textit{null}(C).$$

$\lambda = -1$: *with eigenvectors* $\{v_i\}_{i=1}^{n_b-m_b}$ *and* $\{v_j\}_{j=n_b-m_b+1}^{n_b}$, *as follows:*

$$v_i = (u_i, 0, b_i, r_i) \quad \text{and} \quad v_j = (0, 0, b_j, r_j), \tag{3.4}$$

where $u_i \in null(B) \neq 0$, $Fu_i + C^T b_i = 0$, $b_j \in null(M)$, r_i *and* r_j *free.*

Proof. The corresponding eigenvalue problem is

$$\begin{pmatrix} F & B^T & C^T & 0 \\ B & 0 & 0 & 0 \\ -C & 0 & M & D^T \\ 0 & 0 & D & 0 \end{pmatrix} \begin{pmatrix} u \\ p \\ b \\ r \end{pmatrix} = \lambda \begin{pmatrix} F & B^T & C^T & 0 \\ 0 & -S_{\text{NS}} & 0 & 0 \\ 0 & 0 & -(M + K_C) & -D^T \\ 0 & 0 & -D & 0 \end{pmatrix} \begin{pmatrix} u \\ p \\ b \\ r \end{pmatrix},$$

where $K_C = CK_1C^T$. The four block rows of the generalized eigenvalue problem can be written as

$$(1 - \lambda)(Fu + B^T p + C^T b) = 0, \tag{3.5}$$

$$Bu = -\lambda S_{\text{NS}}\, p, \tag{3.6}$$

$$(1 + \lambda)(Mb + D^T r) + \lambda CK_1C^T b - Cu = 0, \tag{3.7}$$

$$(1 + \lambda)Db = 0. \tag{3.8}$$

If $\lambda = 1$, (3.5) is automatically satisfied. Equation (3.6) simplifies to:

$$p = -S_{\text{NS}}^{-1} Bu.$$

From (3.8) we have $Db = 0$, hence, $b \in \text{null}(D)$. Let us take $r = 0$, then (3.7) yields

$$Cu = (2M + CK_1C^T)b. \tag{3.9}$$

Case 1: Consider $b \in \text{null}(D)$ and $b \neq 0$, then $Cu = (2M + CK_1C^T)b$. Since, rank of C and $(2M + CK_1C^T)$ is $n_b - m_b$, then the condition (3.9) has at least $n_b - m_b$ linearly independent eigenvectors.

Case 2: Consider $b = 0$, then we have that $Cu = 0$. Hence, u must be in the null space of C. Since

$$\dim(\text{null}(C)) \;=\; n_u - n_b + m_b,$$

this accounts for $n_u - n_b + m_b$ such eigenvectors.

Therefore $\lambda = 1$ is an eigenvalue with algebraic multiplicity at least n_u.

If $\lambda = -1$, (3.8) is satisfied, hence, r is free. Simplifying (3.7) obtains

$$CK_1C^T b + Cu = 0. \tag{3.10}$$

Let us take $u \in \text{null}(B)$, then $p = 0$ and the condition for b is

$$Fu + C^T b = 0. \tag{3.11}$$

Under the condition that $u \in \text{null}(B)$, (3.11) satisfies the equality (3.10).

Case 1: Consider $u \in \text{null}(B)$ and $u \neq 0$, then from (3.11) we have $u = -F^{-1}C^T b$. Since the rank of C^T is $n_b - m_b$ and F is full rank, then there are only $n_b - m_b$ such linearly independent b's that determine u. Hence, for this case we obtain at least $n_b - m_b$ such eigenvectors.

Case 2: Consider $u = 0$, then for (3.11) to hold $C^T b = 0$. Therefore, we take $b \in \text{null}(C^T)$. Since, the null space of C^T is made up of discrete gradients then

$$\dim(\text{null}(C^T)) \; = \; m_b.$$

This accounts for m_b such eigenvectors.

Therefore $\lambda = -1$ is an eigenvalue with algebraic multiplicity at least n_b.

□

REMARK 3.2. *Note that in* (3.4)*,* $\{u_i\}$ *is a* subset *of the null vectors of* B*.*

An approximation of the fluid Schur complement, S_{NS}, is needed to create a practical preconditioner. For S_{NS} we will use the pressure-convection diffusion (PCD) preconditioner developed in [3]. The approximation is based on

$$S_{\text{NS}} = BF^{-1}B^T \approx A_p \, F_p^{-1} Q_p, \tag{3.12}$$

where the matrix A_p is the pressure Laplacian, F_p is the pressure convection-diffusion operator and Q_p is the pressure mass matrix.

The application of the preconditioner involves solving systems that require matrix vector products with:

$$\begin{pmatrix} M_X^{-1} & GL^{-1} \\ L^{-1}G^T & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} F & B^T \\ 0 & A_p \, F_p^{-1} Q_p \end{pmatrix}^{-1}. \tag{3.13}$$

We call this preconditioner $\mathcal{P}_2$. Then the preconditioned matrix yields the eigenvalue plot given in Figure 2.2(b). In practice, a combination of multigrid cycles for the elliptic/parabolic type operators and an the axillary space preconditioner developed in [7] for the curl-curl operators would be desirable to yield a scalable application of the preconditioner.

4. Numerical experiments. In this section, we present preliminary numerical results to illustrate the performance of our preconditioning approaches. We use `FEniCS` [9], a finite element software package, to create the matrix system and `MATLAB` to carry out the numerical solves.

We use the notation: ℓ is the mesh level, DoF is the total degrees of freedom, $\text{time}_{\mathcal{P}_1}$ is the solve time with the approximate inverse preconditioner $\mathcal{P}_1$, $\text{it}_{\mathcal{P}_1}$ is the number of GMRES iterations using $\mathcal{P}_1$, $\text{time}_{\mathcal{P}_2}$ is the solve time with the block triangular preconditioner $\mathcal{P}_2$, $\text{it}_{\mathcal{P}_2}$ is the number of GMRES iterations using $\mathcal{P}_2$ and - when we stopped program due to time and memory constraints.

We will consider 2-dimensional test problems: a smooth solution on a convex domain (described in [12, Section 4.5]) and a singular solution on a non-convex domain (described in [5, Section 5.2]). The results are shown in Table 4.1 and 4.2 for the smooth and singular solutions, respectively.

The results in Table 4.1 and 4.2 show very good scalability with respect to the mesh size when we consider S_{NS} to be the exact fluid Schur complement. However, when we introduce the PCD approximation for S_{NS} we start to see a deterioration in scalability. For the smooth solution case, Table 4.1, the iterations are only increasing by one per mesh level (for $\ell > 6$). However, with the singular solution, Table 4.2, there is an increase of between 2 and 3 iterations for the higher levels. We speculate that this behavior appears to be linked to the approximation to the fluid Schur complement and not necessarily to the quality of our formula. This is manifested in particular with the more challenging singular problem.

ℓ	DoF	$S_{\text{NS}} = BF^{-1}B^T$ time$_{\mathcal{P}_1}$	it$_{\mathcal{P}_1}$	time$_{\mathcal{P}_2}$	it$_{\mathcal{P}_2}$	$S_{\text{NS}} = A_P F_P^{-1} Q_P$ time$_{\mathcal{P}_1}$	it$_{\mathcal{P}_1}$	time$_{\mathcal{P}_2}$	it$_{\mathcal{P}_2}$
4	3,108	1.59e-02	3	2.62e-02	6	4.73e-02	11	4.41e-02	13
5	12,868	8.19e-02	3	6.76e-02	6	1.96e-01	13	1.52e-01	15
6	52,356	7.55e-01	2	4.85e-01	5	1.88e+00	14	1.13e+00	17
7	211,204	1.15e+01	2	6.37e+00	5	2.39e+01	16	1.25e+01	19
8	848,388	-	-	-	-	1.47e+02	17	9.90e+01	20
9	3,400,708	-	-	-	-	1.16e+03	18	6.64e+02	21

Table 4.1: Smooth solution: time and iteration results using the exact Schur complement and the PCD Schur complement approximation for the Navier-Stokes subproblem.

ℓ	DoF	$S_{\text{NS}} = BF^{-1}B^T$ time$_{\mathcal{P}_1}$	it$_{\mathcal{P}_1}$	time$_{\mathcal{P}_2}$	it$_{\mathcal{P}_2}$	$S_{\text{NS}} = A_P F_P^{-1} Q_P$ time$_{\mathcal{P}_1}$	it$_{\mathcal{P}_1}$	time$_{\mathcal{P}_2}$	it$_{\mathcal{P}_2}$
4	2,276	1.90e-02	9	1.53e-02	12	6.11e-02	19	3.65e-02	19
5	9,540	1.24e-01	8	1.24e-01	12	2.37e-01	21	1.91e-01	24
6	39,044	9.77e-01	7	5.75e-01	11	1.51e+00	24	7.58e-01	26
7	157,956	9.50e+00	6	7.56e+00	11	1.54e+01	25	8.50e+00	28
8	635,396	-	-	-	-	1.12e+02	27	5.65e+01	29
9	2,548,740	-	-	-	-	7.68e+02	29	4.84e+02	32

Table 4.2: Singular solution: time and iteration results using the exact Schur complement and the PCD Schur complement approximation for the Navier-Stokes subproblem.

Since we see very strong clustering of the eigenvalues around 1 for the approximate inverse preconditioner (Figure 2.2a) and ± 1 for the block triangular preconditioner (Figure 2.2b), we would expect to see approximately a factor of 2 difference between the iteration counts. This is what we observe for the exact Schur complement case. However, when we use the PCD approximation the iteration numbers seem to be roughly the same.

Finally, we look at the timings. Due to the more complex nature of the approximate inverse preconditioner $\mathcal{P}_1$, we see that the timing results are higher than for $\mathcal{P}_2$. However, there only seems to be a maximum of a factor of two difference between the two preconditioners. This factor seems to decrease when the solution requires more iterations to converge, as in Table 4.2.

5. Conclusion. We have introduced new block preconditioning techniques for the MHD model (1.1)–(1.2). Our aim was to develop indefinite preconditioning approaches that utilizes the Maxwell maximal nullity result in [4] and adapt it to the MHD model.

Using the results in [4], we are able to find exact expressions for both a block Schur complement of the MHD model and its inverse. Using this Schur complement we presented a block triangular preconditioner as well as an approximate inverse preconditioner.

The preliminary numerical experiments demonstrate the viability and effectiveness of our approach. Plans for future work include further optimizing our code, utilizing inexact solves to reduce the overall computational work, and applying our preconditioners to three-dimensional problems.

For the class of approximate inverse preconditioners, it is possible to obtain the nice property that the real part of the eigenvalues for the preconditioned matrix are all positive, in contrast to the case of typical block diagonal/triangular preconditioners. If the preconditioned matrix were symmetrizable by a similarity transformation, then this positivity property may be exploitable by using variations of positive definite solvers. This is left as an area for future work.

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