FAST STRUCTURED JACOBI-JACOBI TRANSFORM

JIE SHEN AND YINGWEI WANG

ABSTRACT. A fast structured algorithm for Jacobi-Jacobi transforms is developed in this paper. The algorithm is based on two main ingredients. (i) Derive explicit formulas for connection matrices of two Jacobi expansions with arbitrary indices. (ii) Explore analytically or numerically a low-rank property hidden in the connection matrices, and construct rank structured approximations for them. Combining these two ingredients, we develop a fast structured Jacobi-Jacobi transform with nearly linear complexity (after a one-time precomputation step) between coefficients of two Jacobi expansions with arbitrary indices. An important byproduct of the fast Jacobi-Jacobi transform is the fast Jacobi transform between the function values at a set of Chebyshev-Gauss type points and coefficients of the Jacobi expansion with arbitrary indices. Ample numerical results are presented to illustrate the computational efficiency and accuracy of our algorithm.

Keywords. Jacobi-Jacobi transform, Chebyshev-Jacobi transform, Legendre transform, low-rank property, structured matrices.

AMS subject classifications. 65T50, 65D05, 65N35, 65F30.

1. Introduction

Jacobi polynomials have found applications in many areas of mathematics and applied sciences, notably the approximation theory [6,7], the resolution of Gibbs' phenomenon [5], electrocardiogram data compression [18], and spectral methods for numerical partial differential equations [3,4,14,15]. See also [10,19] which include extended lists of related work. Many applications require transforms between coefficients of Jacobi expansions and values at Jacobi-Gauss type points, and/or between coefficients of Jacobi expansions with different indices. Hence, it is highly desirable to develop algorithms which can perform these transforms as quickly and accurately as possible.

Given $f(x) \in \mathbb{P}_N = \{\text{polynomials with degree equal or less than } N\}$, the Jacobi expansion is of the form

$$f(x) = \sum_{n=0}^{N} f_n^{\alpha,\beta} J_n^{\alpha,\beta}(x), \quad x \in [-1,1], \tag{1.1}$$

where $\{J_n^{\alpha,\beta}\}_{n=0}^N$ are the Jacobi polynomials with indices $\alpha, \beta > -1$. Let $\{x_j \in [-1,1]\}_{0 \le j \le N}$ be a set of collocation points. One often needs to determine the expansion coefficients $(f_n^{\alpha,\beta})_{n=0}^N$ from function values $(f(x_j))_{j=0}^N$ or vice versa, i.e., we need to perform the *forward* and *backward* Jacobi transforms $\mathbf{f}^{\alpha,\beta} = \boldsymbol{T}_f \bar{\mathbf{f}}, \bar{\mathbf{f}} = \boldsymbol{T}_b \mathbf{f}^{\alpha,\beta}$, respectively, where $\bar{\mathbf{f}} = (f(x_j))_{j=0}^N$ and $\mathbf{f}^{\alpha,\beta} = (f_n^{\alpha,\beta})_{n=0}^N$.

Chebyshev polynomials, as a special case of Jacobi polynomials with $\alpha = \beta = -\frac{1}{2}$, are often used because of their near optimal approximation properties and availability of fast transforms thanks to their close relations to Fourier series [2, 9, 13]. More precisely, if $\{x_j\}_{j=0}^N$ are a set of Chebyshev-Gauss-type points, then the transform between the function values $\{f(x_j)\}$ and the expansion coefficients $\{f_n^T\}$ in terms of Chebyshev polynomials $f(x) = \sum_{n=0}^N f_n^T T_n(x), x \in [-1,1]$, can be done by the fast Fourier transform (FFT) in $O(N\log N)$ operations. Unfortunately, such fast algorithms are not available for transforms related to Jacobi polynomials with arbitrary indices. However, if we fix $\{x_j\}_{j=0}^N$ to be a set of Chebyshev-Gauss-type points, then to obtain the coefficients of the Jacobi expansion $(f_n^{\alpha,\beta})_{n=0}^N$, we can proceed in two steps:

- First, we obtain coefficients of the Chebyshev expansion $(f_n^T)_{n=0}^N$, which can be obtained in $O(N\log N)$ operations by FFT;
- Second, we determine $(f_n^{\alpha,\beta})_{n=0}^N$ from $(f_n^T)_{n=0}^N$ through the identity

$$f(x)=\sum_{n=0}^N f_n^T T_n(x)=\sum_{n=0}^N f_n^J J_n^{\alpha,\beta}(x),\quad x\in[-1,1]. \tag{1.2}$$

Using the orthogonal properties of Chebyshev and Jacobi polynomials, one can easily determine *connection matrices* $\boldsymbol{K}^{T\to(\alpha,\beta)}$ and $\boldsymbol{K}^{(\alpha,\beta)\to T}$ such that

$$\mathbf{f}^{\alpha,\beta}=\boldsymbol{K}^{T\to(\alpha,\beta)}\mathbf{f}^T,\quad \mathbf{f}^T=\boldsymbol{K}^{(\alpha,\beta)\to T}\mathbf{f}^{\alpha,\beta}. \tag{1.3}$$

However, $\boldsymbol{K}^{T\to(\alpha,\beta)}$ and $\boldsymbol{K}^{(\alpha,\beta)\to T}$ are full (upper triangular) matrices so that a direct Chebyshev-Jacobi transform will cost $O(N^2)$. The main question we want to address in this paper is how to quickly and accurately compute Chebyshev-Jacobi transforms and more general Jacobi-Jacobi transforms.

The main goal of this paper is to develop fast algorithms, with nearly linear complexity after a one-time precomputation step, for the Chebyshev-Jacobi and Jacobi-Jacobi transforms with arbitrary Jacobi indices. Our method is based on exploring a so-called *low-rank property* of the connection matrices, i.e., their appropriate off-diagonal blocks have small and nearly bounded (numerical) ranks. A useful feature for matrices with the low-rank property is that they can be approximated by rank structured matrices in *hierarchically semiseparable* (HSS) forms [22,23]. We can use existing algorithms in [21, 23] to quickly construct such HSS forms in a precomputation. For a pre-specified accuracy, such a construction takes $O(rN^2)$ flops based on the explicit matrix form, where r is the maximum off-diagonal numerical rank. The construction may also be done via randomization and takes $O(r^2N)$ flops together with $O(r)$ matrix-vector multiplications. After this precomputation, it only costs $O(rN)$ flops to perform the desired transforms.

The remaining sections are organized as follows. In Section 2, we derive explicit recurrence formulae of the connection matrices for Jacobi-Jacobi transforms with arbitrary indices. In Section 3, we explore the low-rank property of Jacobi-Jacobi connection matrices, and briefly mention the HSS construction. Several numerical experiments of the proposed fast structured Jacobi transforms are shown in Section 4.

2. Connection coefficients for Jacobi-Jacobi transforms

We consider the transform between the coefficients of two Jacobi expansions with different indices, which is the generalization of the *forward* and *backward Chebyshev-Jacobi transforms* (FCJT and BCJT) shown in (1.3).

Let us consider the following two Jacobi expansions:

$$f(x)=\sum_{n=0}^N f_n^1 J_n^{\alpha_1,\beta_1}(x),=\sum_{n=0}^N f_n^2 J_n^{\alpha_2,\beta_2}(x),\quad x\in[-1,1], \tag{2.1}$$

where the indices (α_1, β_1) and (α_2, β_2) can be any real number bigger than -1. The connection coefficients between the above two Jacobi polynomials satisfy the following relations:

$$\mathbf{f}^1 = \boldsymbol{K}^{2\to 1}\mathbf{f}^2, \quad \mathbf{f}^2 = \boldsymbol{K}^{1\to 2}\mathbf{f}^1, \tag{2.2}$$

About the entries of the above two connection coefficients, we have the following theorem, the proof of which is omitted here for shortness.

Theorem 2.1 (Recurrence formula for Jacobi-Jacobi transform)**.** *The nonzero entries of* $\boldsymbol{K}^{2\to 1} = (\kappa_{i,j}^{2\to 1})_{i,j=0}^N$ *and* $\boldsymbol{K}^{1\to 2} = (\kappa_{i,j}^{1\to 2})_{i,j=0}^N$ *defined in* (2.2) *can be generated recursively as follows:*

$$\kappa_{i,j+1}^{2\to 1} = \varepsilon_1^{2\to 1}\kappa_{i,j-1}^{2\to 1} + \varepsilon_2^{2\to 1}\kappa_{i-1,j}^{2\to 1} + \varepsilon_3^{2\to 1}\kappa_{i,j}^{2\to 1} + \varepsilon_4^{2\to 1}\kappa_{i+1,j}^{2\to 1}, \quad j \geq i, \tag{2.3}$$

$$\kappa_{i,j+1}^{1\to 2} = \varepsilon_1^{1\to 2}\kappa_{i,j-1}^{1\to 2} + \varepsilon_2^{1\to 2}\kappa_{i-1,j}^{1\to 2} + \varepsilon_3^{1\to 2}\kappa_{i,j}^{1\to 2} + \varepsilon_4^{1\to 2}\kappa_{i+1,j}^{1\to 2}, \quad j \geq i, \tag{2.4}$$

where the coefficients $\{\varepsilon_k^{m\to\delta(m)}\}_{k=1,2,3,4}^{m=1,2}$ *(with* $\delta(1) = 2$ *and* $\delta(2) = 1$*) are given by*

$$\varepsilon_1^{m\to\delta(m)} = -r_j^m, \qquad \varepsilon_2^{m\to\delta(m)} = \begin{cases} 0, & i = 0, \\ \pi_2^{\delta(m)}(i)p_j^m, & i \geq 1, \end{cases}$$

$$\varepsilon_3^{m\to\delta(m)} = \pi_3^{\delta(m)}(i)p_j^m - q_j^m, \qquad \varepsilon_4^{m\to\delta(m)} = \pi_4^{\delta(m)}(i)p_j^m,$$

with the parameters

$$p_n^{\alpha,\beta} = \frac{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}{2(n+1)(n+\alpha+\beta+1)}, \qquad q_n^{\alpha,\beta} = \frac{(\beta^2-\alpha^2)(2n+\alpha+\beta+1)}{2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)},$$

$$r_n^{\alpha,\beta} = \frac{(n+\alpha)(n+\beta)(2n+\alpha+\beta+2)}{(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)}, \quad \pi_2^m(k) = \frac{2k(k+\alpha_m+\beta_m)}{(2k+\alpha_m+\beta_m-1)(2k+\alpha_m+\beta_m)},$$

$$\pi_3^m(k) = \frac{\beta_m^2-\alpha_m^2}{(2k+\alpha_m+\beta_m)(2k+\alpha_m+\beta_m+2)}, \quad \pi_4^m(k) = \frac{2(k+\alpha_m+1)(k+\beta_m+1)}{(2k+\alpha_m+\beta_m+2)(2k+\alpha_m+\beta_m+3)},$$

for $m = 1, 2$ *and* $k = 0, 1, 2, \ldots$.

Moreover, the starting points of the above recurrence formula are

$$\kappa_{0,0}^{2\to 1} = 1, \quad \kappa_{0,1}^{2\to 1} = \frac{(\beta_1-\alpha_1)(\alpha_2+\beta_2+2)}{2(\alpha_1+\beta_1+2)} - \frac{\beta_2-\alpha_2}{2}, \quad \kappa_{1,0}^{2\to 1} = 0, \quad \kappa_{1,1}^{2\to 1} = \frac{\alpha_2+\beta_2+2}{\alpha_1+\beta_1+2},$$

$$\kappa_{0,0}^{1\to 2} = 1, \quad \kappa_{0,1}^{1\to 2} = \frac{(\alpha_1+\beta_1+2)(\beta_2-\alpha_2)}{2(\alpha_2+\beta_2+2)} - \frac{\beta_1-\alpha_1}{2}, \quad \kappa_{1,0}^{1\to 2} = 0, \quad \kappa_{1,1}^{1\to 2} = \frac{\alpha_1+\beta_1+2}{\alpha_2+\beta_2+2}.$$

Now let us consider the most useful case of Chebyshev-Jacobi transforms, i.e., the connection problem between the Chebyshev expansion and the Jacobi expansion, shown in (1.2). Note that the Chebyshev polynomials and Jacobi polynomials with indices $\alpha = \beta = -1/2$ are proportional to each other, i.e. $T_n(x) \equiv \frac{\Gamma(1/2)\Gamma(n+1)}{\Gamma(n+1/2)} J_n^{-1/2,-1/2}(x), \quad \forall n = 0, 1, \ldots$. It implies that we cannot simply take $\alpha_1 = \beta_1 = \frac{1}{2}$ and $\alpha_2 = \alpha, \beta_1 = \beta$ to get the recurrence relation for the entries of the connection matrices $\boldsymbol{K}^{T\to(\alpha,\beta)}$ and $\boldsymbol{K}^{(\alpha,\beta)\to T}$ defined in (1.3). Instead, we can prove the following theorem.

Theorem 2.2 (Recurrence formula for Chebyshev-Jacobi transform)**.** *Denote* $J = (\alpha, \beta)$*. The nonzero entries of* $\boldsymbol{K}^{J\to T} = (\kappa_{ij}^{J\to T})_{i,j=0}^N$ *and* $\boldsymbol{K}^{T\to J} = (\kappa_{ij}^{T\to J})_{i,j=0}^N$ *can be generated recursively as* *follows:*

$$\kappa_{i,j+1}^{J\to T} = \varepsilon_1^{J\to T}\kappa_{i,j-1}^{J\to T} + \varepsilon_2^{J\to T}\kappa_{i-1,j}^{J\to T} + \varepsilon_3^{J\to T}\kappa_{i,j}^{J\to T} + \varepsilon_4^{J\to T}\kappa_{i+1,j}^{J\to T}, \quad j \ge i, \tag{2.5}$$

$$\kappa_{i,j+1}^{T\to J} = \varepsilon_1^{T\to J}\kappa_{i,j-1}^{T\to J} + \varepsilon_2^{T\to J}\kappa_{i-1,j}^{T\to J} + \varepsilon_3^{T\to J}\kappa_{i,j}^{T\to J} + \varepsilon_4^{T\to J}\kappa_{i+1,j}^{T\to J}, \quad j \ge i, \tag{2.6}$$

where the parameters $\varepsilon_k^{J\to T}$ *and* $\varepsilon_k^{T\to J}$*, for* $k = 1, 2, 3, 4$*, are*

$$\varepsilon_1^{J\to T} = -r_j^{\alpha,\beta}, \quad \varepsilon_2^{J\to T} = \begin{cases} 0, & i = 0, \\ p_j^{\alpha,\beta}, & i = 1, \\ \frac{1}{2}p_j^{\alpha,\beta}, & i \ge 2, \end{cases} \quad \varepsilon_3^{J\to T} = -q_j^{\alpha,\beta}, \quad \varepsilon_4^{J\to T} = \begin{cases} \frac{1}{4}p_j^{\alpha,\beta}, & i = 0, \\ \frac{1}{2}p_j^{\alpha,\beta}, & i \ge 1, \end{cases}$$

with the parameters $\{p_j^{\alpha,\beta}, q_j^{\alpha,\beta}, r_j^{\alpha,\beta}\}$ *are the same as those in Theorem 2.1 and*

$$\varepsilon_1^{T\to J} = -1, \qquad \varepsilon_2^{T\to J} = \begin{cases} 0, & i = 0, \\ \frac{4i(\alpha+\beta+i)}{(\alpha+\beta+2i-1)(\alpha+\beta+2i)}, & i \ge 1, \end{cases}$$

$$\varepsilon_3^{T\to J} = \begin{cases} -\frac{2(\alpha-\beta)}{\alpha+\beta+2}, & i = 0, \\ \frac{2(\beta^2-\alpha^2)}{(\alpha+\beta+2i)(\alpha+\beta+2i+2)}, & i \ge 1, \end{cases} \qquad \varepsilon_4^{T\to J} = \frac{4(\alpha+i+1)(\beta+i+1)}{(\alpha+\beta+2i+2)(\alpha+\beta+2i+3)}.$$

Moreover, the starting points of the above recurrence relation are

$$\kappa_{00}^{J\to T} = 1, \qquad \kappa_{01}^{J\to T} = \frac{\alpha-\beta}{2}, \qquad \kappa_{10}^{J\to T} = 0, \qquad \kappa_{11}^{J\to T} = \frac{\alpha+\beta+2}{2},$$

$$\kappa_{00}^{T\to J} = 1, \qquad \kappa_{01}^{T\to J} = -\frac{\alpha-\beta}{\alpha+\beta+2}, \qquad \kappa_{10}^{T\to J} = 0, \qquad \kappa_{11}^{T\to J} = \frac{2}{\alpha+\beta+2}.$$

Remark 2.1. *Thanks to the symmetry property of Jacobi polynomials* $J_n^{\alpha,\beta}(-x) = (-1)^n J_n^{\beta,\alpha}(x)$*, the Jacobi polynomial* $J_n^{\alpha,\alpha}(x)$ *(up to a constant, referred to as the Gegenbauer or ultra-spherical polynomial), is an odd function for odd* n *and an even function for even* n*. Therefore, we have* $\kappa_{ij}^{T\to J} = \kappa_{ij}^{T\to J} = 0$ *for* $i+j$ *odd, if* $\alpha = \beta$*.*

Remark 2.2. *For the problem between Jacobi polynomials with integer differences, i.e., both of* $|\alpha_1 - \alpha_2|$ *and* $|\beta_1 - \beta_2|$ *in (2.1) are integers, one can find that the matrix itself can be written as the product of banded or separable matrices, which means that the Jacobi-Jacobi transform can be done in linear complexity.*

3. Low-rank property and HSS structures

In this section, we show that the Jacobi-Jacobi connection matrices given in Section 2 enjoy a so-called *low-rank property*, which allows us to construct hierarchically semiseparable (HSS) approximations to the matrices [23]. HSS representations provide an efficient and stable way to explore the rank structures of a matrix A. That is, if the off-diagonal blocks of A have small ranks or numerical ranks, then we say A has a low-rank property, and can rewrite or approximation A by an HSS form. Such a form is data-sparse in the sense that those dense off-diagonal blocks are in compressed low-rank format. This helps to significantly reduce the algorithmic complexity and storage for handling A. In particular, if A is of order N and its largest off-diagonal rank or numerical rank is r, then the multiplication of A and a vector costs only $O(rN)$ flops. In the following, we show that r is very small, so that the HSS Jacobi-Jacobi transformation is very efficient.

Let us start with the plots of the numerical ranks of HSS block rows of the connection coefficients $\boldsymbol{K}^{(\alpha^*,\beta^*)\to(\alpha,\beta)}$ and $\boldsymbol{K}^{(\alpha,\beta)\to(\alpha^*,\beta^*)}$ for (α,β) in different regions with centers (α^*,β^*). Each matrix is hierarchically partitioned into $l_{\max}$ levels of HSS blocks, which are block rows or columns excluding the diagonal subblocks [23]. At level $l=0,1,\ldots,l_{\max}$, the HSS block rows have row sizes $N_l = n/2^l$ and maximum numerical rank r_l. We will see that when l decreases, N_l doubles, but r_l only increases slightly, which is independent the location of (α^*,β^*). We consider (α,β) in three regions: (i) $\Omega_I = [-1,0]^2$ with center $(-1/2,-1/2)$, which corresponds to the Chebyshev-Jacobi case; (ii) $\Omega_{II} = [(\alpha^*-1/2,\beta^*-1/2)]^2$ with $\alpha^* = 3\sqrt{3}, \beta^* = \pi$; (iii) $\Omega_{III} = [(\alpha^*-4,\beta^*-4)]^2\backslash[(\alpha^*-3,\beta^*-3)]^2$, with the same center (α^*,β^*).

In practice, we randomly choose 40 points in the square regions Ω_I and Ω_{II} and 100 points in the square-ring region Ω_{III}. In particular, we add one more point $(\alpha,\beta)=(0,0)$ in region Ω_I, which corresponds to the Chebyshev-Legendre case. See Figure 3.1 for a graphical representation of these regions. In Figure 3.2, we show r_l (versus N_l) for the HSS block rows at level l of the HSS partition, where the relative tolerance for computing the numerical ranks is $\tau = 10^{-8}$, the bottom level HSS block row size is $N_{l_{\max}} = 20$. For comparison purposes, we also plot shifted $\log(N_l)$ and $\log\log(N_l)$ curves. We can observe that the following.

(1) In all of the three cases, the numerical HSS ranks r_l increases very slowly, in fact, much slower than $O(\log N_l)$. Instead, it roughly follows the pattern of $O(\log\log N_l)$ in our computation, although not yet analytically justified. This observation is useful for the derivation of the nearly linear complexity of our HSS construction.
(2) The numerical HSS ranks between two sets of indices (α,β) and (α^*,β^*) appear to depend only on the distance between (α,β) and (α^*,β^*), and are nearly independent of their relative locations, by the comparison of the results from Ω_I and Ω_{II} shown in the first four parts of Figure 3.2.
(3) As the distance between (α,β) and (α^*,β^*) increases, the HSS ranks decrease, which is shown in the last two parts of Figure 3.2.

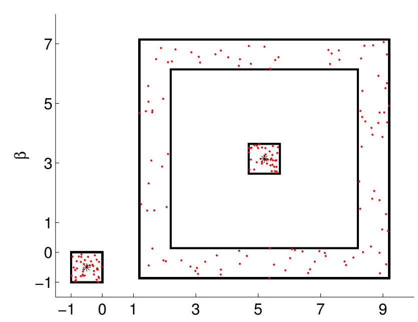
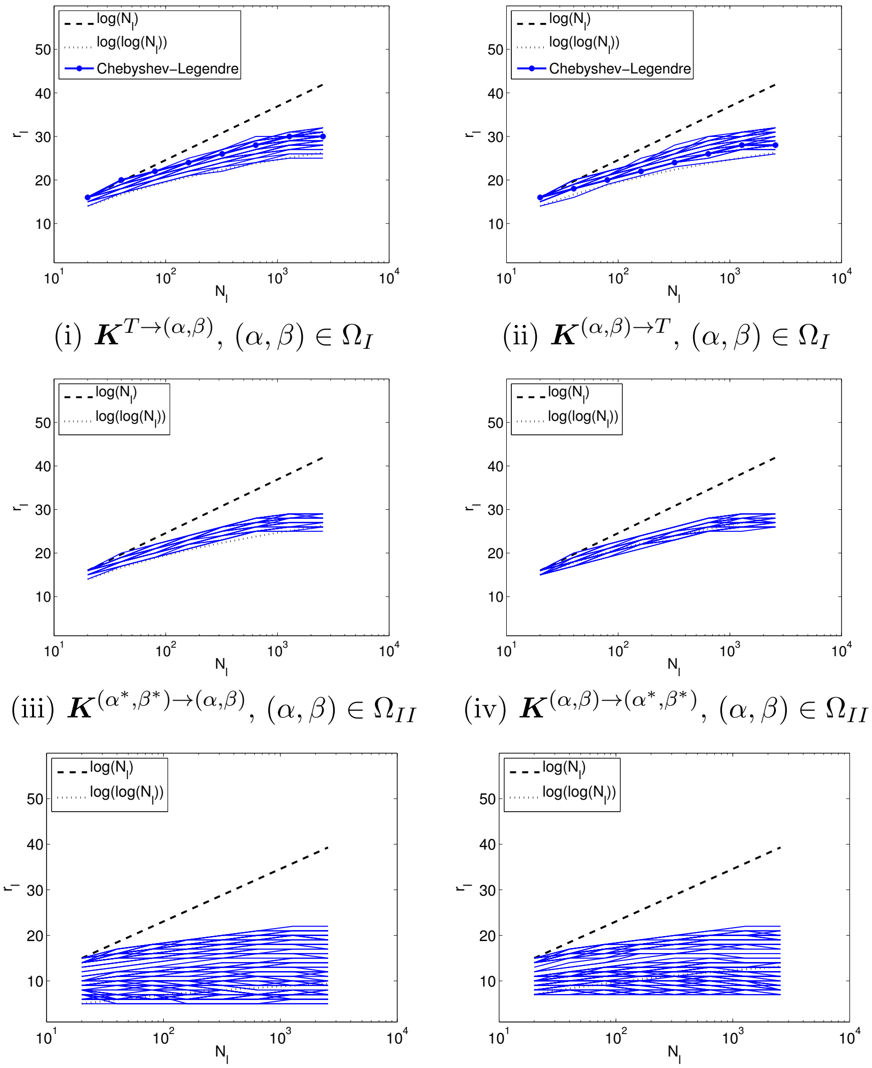


Figure 3.1: *Jacobi indices (α,β) tested later in Figures 3.2, where the left square (Ω_I) is centered at $(-1/2,-1/2)$, and the right squares (inner square Ω_{II} and outer banded region Ω_{III}) are centered at $(3\sqrt{3},\pi)$.*



(i) $\boldsymbol{K}^{T\to(\alpha,\beta)}$, $(\alpha,\beta)\in\Omega_I$ (ii) $\boldsymbol{K}^{(\alpha,\beta)\to T}$, $(\alpha,\beta)\in\Omega_I$

(iii) $\boldsymbol{K}^{(\alpha^*,\beta^*)\to(\alpha,\beta)}$, $(\alpha,\beta)\in\Omega_{II}$ (iv) $\boldsymbol{K}^{(\alpha,\beta)\to(\alpha^*,\beta^*)}$, $(\alpha,\beta)\in\Omega_{II}$

(v) $\boldsymbol{K}^{(\alpha^*,\beta^*)\to(\alpha,\beta)}$, $(\alpha,\beta)\in\Omega_{III}$ (vi) $\boldsymbol{K}^{(\alpha,\beta)\to(\alpha^*,\beta^*)}$, $(\alpha,\beta)\in\Omega_{III}$

Figure 3.2: *Numerical HSS ranks of Chebyshev-Jacobi transforms.*

4. Fast structured Jacobi-Jacobi transforms

We now present some numerical experiments to illustrate the efficiency and accuracy of our fast structured transforms. All the tests are carried out on a Thinkpad T430s laptop with 4GB RAM and an Intel i7 core at 2.9GHz. We first mention some remarks about the experiments: (i) N means the size of the matrices; (ii) $\mathbf{t}_c$ is the CPU time of HSS construction, $\mathbf{e}_c=\frac{\|\boldsymbol{A}-\tilde{\boldsymbol{A}}\|_2}{\|\boldsymbol{A}\|_2}$ is the relative error of the HSS approximation to matrix $\boldsymbol{A}$; (iii) The tolerance in the HSS construction is chosen as $\tau = 10^{-12}$ and the row sizes of the finest level HSS block rows are about 40. Here, τ is a relative tolerance which is different from that used in the theorems above.

4.1. **Jacobi-Jacobi transforms.** Let us start with the Cheyshev-Legendre transform, which is the most useful case among Jacobi-Jacobi transforms. Hale and Townsend [8] proposed a fast algorithm for Cheyshev-Legendre transforms using an asymptotic formula (CLTAF), with computational complexity $O(N \log^2 N/\log\log N)$. We will compare our proposed fast structured Chebyshev-Legendre transform (FSCLT) with CLTAF as well as the direct Chebyshev-Legendre transforms (DCLT).

(1) *HSS construction.* The computational cost and approximation errors of the HSS construction for forward and backward Chebyshev-Legendre connection matrices are shown in Table 1. We observe that the complexity of the HSS construction scales roughly as $O(N^2)$.
(2) *Chebyshev-Legendre transform.* We randomly choose a vector $\mathbf{v}$ of length N, and then perform the forward and backward Chebyshev-Legendre transform on $\mathbf{v}$ for 100 times. The CPU times of DCLT, CLTAF and FSCLT, as well as $O(N)$ and $O(N^2)$ reference lines are shown in Figure 4.1. For the case with small N, there is no significant difference between these three methods. As N increases, the cost of DCLT grows like $O(N^2)$, and the growth rate of CLTAF is between $O(N^2)$ and $O(N)$ (likely $O(N(\log N)^2/\log\log N)$ as claimed in [8]), while our FSCLT is of nearly linear complexity, more precisely, $O(rN)$, where r is abound $O(\log\log N)$.

Now let us consider the Jacobi-Jacobi transform between (α^*, β^*) and (α', β'). On one hand, we choose $\alpha^* = 3\sqrt{3}, \beta^* = \pi$, because we want to verify that our algorithms work for any real number bigger than -1. On the other hand, we consider two cases for (α', β'): (i) *non-integer differences*, $\alpha'_1 = 2, \beta'_1 = 1$; (ii) *integer differences* $\alpha'_2 = 3\sqrt{3} + 2, \beta'_2 = \pi + 1$. For the case (i), we compare the results between direct Jacobi-Jacobi transform (DJJT) and our fast structured Jacobi-Jacobi transform (FSJJT). For the case (ii), we show the results for the Jacobi-Jacobi promotion transform (JJPT) and the Jacobi-Jacobi demotion transform (JJDT). The computational time is shown in Figure 4.2. We can observe that

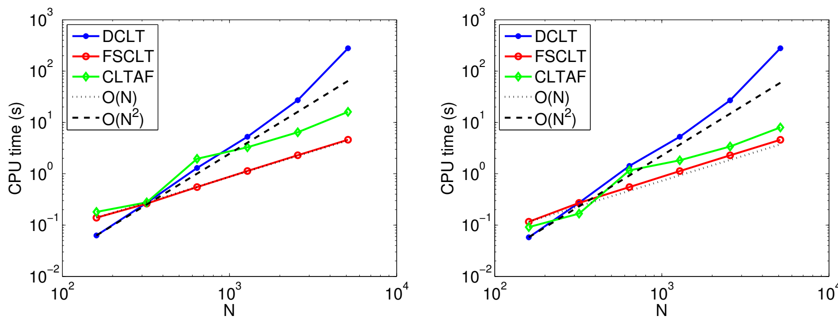
(1) The Jacobi-Jacobi transforms for indices with integer differences are very fast with almost linear complexity, for both promotion and demotion.
(2) The cost of our proposed FSJJT is almost linear in N, while the direct transform is quadratic.

Table 1: *HSS construction cost and accuracy for Chebyshev-Legendre transforms.*

N	$\boldsymbol{K}^{T\to L}$		$\boldsymbol{K}^{L\to T}$	
	$\mathbf{t}_c$	$\mathbf{e}_c$	$\mathbf{t}_c$	$\mathbf{e}_c$
160	0.080	2.2984e-13	0.005	2.4553e-13
320	0.016	1.1482e-12	0.018	3.2789e-13
640	0.067	1.3069e-12	0.060	3.6378e-13
1280	0.266	1.5614e-12	0.243	1.0479e-12
2560	1.230	2.3320e-12	1.048	1.8469e-12
5120	4.828	1.0110e-11	5.344	2.3042e-12

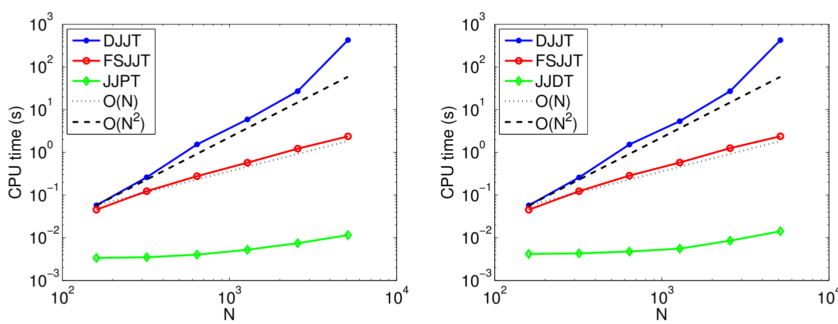
Table 2: *Errors for Chebyshev-Legendre transforms.*

N	$\boldsymbol{K}^{T\to L}$		$\boldsymbol{K}^{L\to T}$	
	FSCLT	CLTAT [8]	FSCLT	CLTAT [8]
160	6.1466e-14	1.5815e-14	4.3208e-14	2.1671e-14
320	2.7861e-13	3.1610e-14	3.0375e-13	1.5178e-13
640	2.2266e-13	2.7652e-14	3.0139e-13	8.5224e-13
1280	1.7845e-13	5.3438e-14	5.7378e-13	7.1993e-13
2560	2.2830e-13	9.9047e-14	1.5104e-12	1.3362e-11
5120	4.1844e-13	1.1875e-13	1.2649e-12	1.3817e-11



(i) $\boldsymbol{K}^{T\to L}$ (ii) $\boldsymbol{K}^{L\to T}$

Figure 4.1: *CPU time of forward and backward Chebyshev-Legendre transforms.*



(i) $\boldsymbol{K}^{(\alpha^*,\beta^*)\to(\alpha',\beta')}$ (ii) $\boldsymbol{K}^{(\alpha',\beta')\to(\alpha^*,\beta^*)}$

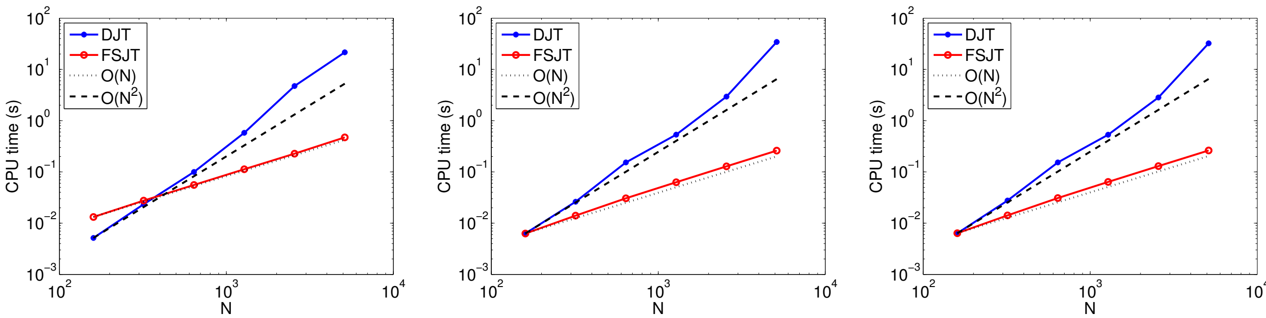
Figure 4.2: *CPU time of Jacobi-Jacobi connection problems*

4.2. **Jacobi transforms.** We then consider the functions $f_k(t)$ defined by

$$f_k(t) = |t - \sin(k)|, \quad k = 1, 2, 3, \dots, \tag{4.1}$$

where the function $f_k(t)$ is only continuous but not differentiable at the points $t_k = \sin k$. This choice is made to ensure that we do not compute expansion coefficients that essentially vanish at moderate degrees. For both functions, we repeat the forward Jacobi transforms from $k = 1$ to $k = 10$, for different degrees N. Besides, the errors shown in the following tables are the average of the 10 computations.

We compare the results of our fast structured Jacobi transform (FSJT) with direct Jacobi transform (DJT) for any indices. The computational time for the cases with $\alpha = \beta = 0$, $\alpha = -\frac{\sqrt{2}}{2}, \beta = \frac{\pi}{4}$ and $\alpha = 10\sqrt{3}, \beta = 10\pi$ are shown Figure 4.3. These results again demonstrate that our prosed FSJT has almost linear complexity, for arbitrary indices α, β.



(i) $\alpha = \beta = 0$ (ii) $\alpha = -\frac{\sqrt{2}}{2}, \beta = \frac{\pi}{4}$ (iii) $\alpha = 10\sqrt{3}, \beta = 10\pi$

Figure 4.3: *CPU time of Jacobi transforms with different indices for function set* $\{f_k\}_{k=1}^{10}$.

5. Concluding remarks

In this paper, we developed efficient and robust algorithms for Jacobi-Jacobi transforms with arbitrary indices. To achieve this, we derived explicit formulae for the connection between two Jacobi polynomials with different indices, and then showed that these matrices have the low-rank property. The key to the success of our method lies at the HSS approximation of the connection matrices. After a one-time HSS construction cost, the Jacobi-Jacobi transforms can be accomplished in nearly linear complexity. To the authors' best knowledge, this is the first algorithm which can perform the Jacobi-Jacobi transforms between two sets of arbitrary indices with nearly linear complexity $O(rN)$ with r behaving like $O(\log N)$ or even better.

The main techniques and strategies developed in this paper can be applied to many other situations. For example, a more difficult problem is to construct a fast spherical harmonic transform. Many attempts have been made in this regard [11,16,17,20], but they are still not fully satisfactory. The main difficulty, as compared with the Jacobi case, is that the spherical harmonic expansion involves associate Legendre polynomials with a full range of indices, rather than a fixed index. It is hopeful that, by exploring the relations between associate Legendre polynomials and Chebyshev polynomials, one can construct a robust and fast spherical harmonic transform.

References

[1] B. Alpert and V. Rokhlin, *A fast algorithm for the evaluation of Legendre expansions*, SIAM Journal on Scientific and Statistical Computing, 12 (1991), pp. 158–179.

[2] J. P. Boyd, *Chebyshev and Fourier Spectral Methods*, Dover Publications Inc., Mineola, NY, 2nd ed., 2001.

[3] C. Canuto, M. Y. Hussaini, A. Quarteroni, and T. A. Zang, *Spectral Methods: Fundamentals in Single Domains*, Springer-Verlag, Berlin-Heidelberg, 2006.

[4] D. Gottlieb and S. A. Orszag, *Numerical Analysis of Spectral Methods: Theory and Applications*, no. 26 in CBMS-NSF Regional Conference Series in Applied Mathematics, SIAM, Philadelphia, 1977.

[5] D. Gottlieb and C. Shu, *On the Gibbs phenomenon and its resolution*, SIAM Review, 39 (1997), pp. 644–668.

[6] B.-Y. Guo, *Jacobi approximations in certain Hilbert spaces and their applications to singular differential equations*, Journal of Mathematical Analysis and Applications, 243 (2000), pp. 373–408.

[7] B.-Y. Guo and L.-L. Wang, *Jacobi approximations in non-uniformly Jacobi-weighted Sobolev spaces*, Journal of Approximation Theory, 128 (2004), pp. 1–41.

[8] N. Hale and A. Townsend, *A fast, simple, and stable Chebyshev-Legendre transform using an asymptotic formula*, SIAM Journal on Scientific Computing, 36 (2014), pp. A148–A167.

[9] J. C. Mason and D. C. Handscomb, *Chebyshev Polynomials*, Taylor & Francis, New York, USA, 2002.

[10] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark, eds., *NIST Handbook of Mathematical Functions*, Cambridge University Press, New York, NY, 2010.

[11] V. Rokhlin and M. Tygert, *Fast algorithms for spherical harmonic expansions*, SIAM Journal on Scientific Computing, 27 (2006), pp. 1903–1928.

[12] J. Shen, *Effcient Chebyshev-Legendre Galerkin methods for elliptic problems*, Proceedings of ICOSAHOM'95, 34 (1995), pp. 233–239.

[13] ———, *Efficient spectral-Galerkin method II: Direct solvers of second- and fourth-order equations using Chebyshev polynomials*, SIAM J. Sci. Comput., 16 (1995), pp. 74–87.

[14] J. Shen and T. Tang, *Spectral and High-Order Methods with Applications*, Science Press of China, Beijing, 2006.

[15] J. Shen, T. Tang, and L.-L. Wang, *Spectral Methods: Algorithms, Analysis and Applications*, no. 41 in Springer Series in Computational Mathematics, Springer-Verlag, Berlin-Heidelberg, 2011.

[16] R. Suda and M. Takami, *A fast spherical harmonics transform algorithm*, Math. Comput., 71 (2002), pp. 703–715.

[17] P. N. Swarztrauber and W. F. Spotz, *Generalized discrete spherical harmonic transforms*, J. Comput. Phys., 159 (2000), pp. 213 – 230.

[18] D. Tchiotsop, D. Wolf, V. Louis-Dorr, and R. Husson, *ECG data compression using Jacobi polynomials*, in Proceedings of the 29th Annual International Conference of the IEEE EMBS, Lyon France, 2007, pp. 1863–1867.

[19] N. M. Temme, *Special Functions: An Introduction to the Classical Functions of Mathematical Physics*, John Wiley & Sons, Inc., New York, 1996.

[20] M. Tygert, *Fast algorithms for spherical harmonic expansions II*, Journal of Computational Physics, 227 (2008), pp. 4260–4279.

[21] Y. Xi, J. Xia, and R. Chan, *A fast randomized eigensolver with structured LDL factorization update*, SIAM J. Matrix Anal. Appl., 35 (2014), pp. 974–996.

[22] J. Xia, *On the complexity of some hierarchical structured matrix algorithms*, SIAM J. Matrix Anal. Appl., 33 (2012), pp. 388–410.

[23] J. Xia, S. Chandrasekaran, M. Gu, and X. S. Li, *Fast algorithms for hierarchically semiseparable matrices*, Numer. Linear Algebra Appl., 17 (2010), pp. 953–976.

Department of Mathematics, Purdue University, West Lafayette, IN 47907

E-mail address: shen7@purdue.edu

Department of Mathematics, Purdue University, West Lafayette, IN 47907

E-mail address: wywshtj@gmail.com

URL: www.math.purdue.edu/∼wang838