Uzawa smoother in multigrid for coupled porous medium and Stokes flow system

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SUMMARY

The multigrid solution of coupled porous media and Stokes flow problems is considered. The Darcy equation as the saturated porous medium model is coupled to the Stokes equations by means of appropriate interface conditions. We focus on an efficient multigrid solution technique for the coupled problem, which is discretized by finite volumes on staggered grids, giving rise to a saddle point linear system. Special treatment is required regarding the discretization at the interface. An Uzawa smoother is employed in multigrid, which is a decoupled procedure based on symmetric Gauss-Seidel smoothing for velocity components and a simple Richardson iteration for the pressure field. Since a relaxation parameter is part of a Richardson iteration, Local Fourier Analysis (LFA) is applied to determine the optimal parameters. Highly satisfactory multigrid convergence is reported, and, moreover, the algorithm performs well for small values of the hydraulic conductivity and fluid viscosity, that are relevant for applications.

KEY WORDS: Darcy equation, Porous medium, Stokes equation, Free flow, Coupling, interface conditions, Multigrid method, Uzawa smoother, local Fourier analysis

1. INTRODUCTION

Coupling of free flow and a saturated porous medium models has received considerable attention due to its application in environmental and industrial context, such as in flood simulation, filtration, contamination, and so on. It is challenging to deal with a coupled system, since each part is based on a different model and an appropriate coupling at the interface is required. Flow in the saturated porous medium is modeled by the conventional Darcy equation here (the solid framework is assumed to be rigid and there is no interaction between the fluid and solid matrix in the porous medium), while the Newtonian flow through a channel is modeled by the incompressible Stokes equations. Appropriate interface conditions are based on the principles of mass conservation, equilibrium of normal stresses across the interface and a special condition called the Beavers-Joseph-Saffman [16, 25] describing the relation between the shear stress and the tangential velocity. The coupled problem is discretized by the finite volume method on a staggered grid, which also results in a symmetric system of linear equations of saddle point form. Many researchers have studied the coupled problem theoretically, see [1, 7, 19, 21]. We focus on an efficient numerical technique for the discrete coupled problem. Related literature can be found in [2, 6, 8, 9, 10, 11, 15, 18, 23, 24].

There are several ways to solve a coupled system. A popular technique is based on domain decomposition (DD) [17, 26]. The main idea is then to update the subdomain problems iteratively, until convergence [6, 9, 11]. In this paper, we consider a monolithic multigrid strategy. The multigrid method is an efficient solution technique for linear and nonlinear systems of equations, and we employ it for solving the discretized Darcy-Stokes coupled problem. The choice of smoother plays an important role on the performance of multigrid. Basically, there are two major categories

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of smoothers for saddle point problems: coupled and decoupled smoothers, see [12, 13, 22]. Coupled smoothers, like the Vanka smoother, or box relaxation, was introduced in [28] for incompressible flow problems and has been often used since then for various saddle point type problems. Decoupled smoothers are popular because of their convenient implementation. In this paper, an equation-wise, decoupled smoother called the Uzawa smoother is taken into consideration for the coupled Darcy/Stokes system.

The discretized Darcy and Stokes problems on staggered grids all have a saddle point form [3], where a zero block appears on the diagonal of the system matrix. The Uzawa smoother, studied in a PhD thesis, as well as in a conference proceedings paper by P. Nigon [20], can be applied for this kind of system. This smoother has been enhanced for the Stokes equations in [14]. For the problem here, the velocities in the Darcy and Stokes equations are updated first, after which the pressures for both subsystems are updated. We deal with a so-called multiblock multigrid algorithm which is decoupled based on the idea of grid partitioning. Boundary updates are communicated between the two blocks within the algorithm on each multigrid level.

The Uzawa smoother is based on a Richardson iteration in which a relaxation parameter occurs. Local Fourier analysis is applied to choose such a suitable relaxation parameter. LFA is a powerful tool for the quantitative analysis of the convergence of multigrid, introduced by Brandt [4] in 1977 and then developed in [5]. A general introduction can be found in [27] and software is available [29]. As the optimal relaxation parameter for the Stokes problem has already been determined in [14], we are concerned with the selection of an optimal parameter for the Darcy problem through LFA in the present paper. Here, LFA is also used to confirm the convergence obtained from the monolithic multigrid method. LFA is applied to both Darcy and Stokes subproblems separately, and it is shown that the worst of these factors results to be the global convergence of the multigrid for the coupled problem.

The paper is organized as follows. The equations in free flow and porous media, together with the interface conditions are introduced in Section 2. Section 3 deals with the discretization of the coupled Darcy-Stokes system. We give the discrete formulas for the coupled system including the discretization at the interface. The solution method, the Uzawa smoother and its analysis by means of LFA, are presented in Section 4. In Section 5, several numerical experiments are performed to show the algorithm's efficiency. Conclusions are drawn in Section 6.

2. PROBLEM FORMULATION

We consider the coupled Darcy/Stokes problem on a bounded domain $\Omega \subset \mathbb{R}^2$. We assume that Ω is subdivided into two disjoint subdomains Ω^d and Ω^f, corresponding to the porous medium and free flow regions, respectively. Let Γ denote the interface between the two subregions, that is, $\Gamma = \partial\Omega^d \cap \partial\Omega^f$. The geometry of the problem is represented in Figure 1, where we also display $\mathbf{n}^f$ and $\mathbf{n}^d$, denoting the unit outward normal vectors on $\partial\Omega^f$ and $\partial\Omega^d$, respectively. Note that $\mathbf{n}^f = -\mathbf{n}^d$ at the interface Γ.

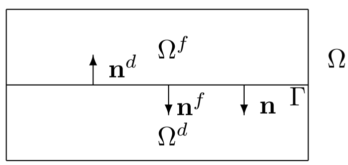


Figure 1. Geometry of the coupled Darcy/Stokes problem. Subdivision of the domain Ω into a free flow subregion Ω^f and a porous medium subdomain Ω^d, by an internal interface Γ.

The fluid flow through a rigid and saturated porous medium Ω^d is described by Darcy's law. The mixed formulation of the Darcy problem reads

$$\begin{aligned} \mathbb{K}^{-1}\mathbf{u}^d + \nabla p^d &= \mathbf{0} \quad \text{in } \Omega^d \,, \\ \nabla \cdot \mathbf{u}^d &= f^d \quad \text{in } \Omega^d \,, \end{aligned} \tag{1}$$

where $\mathbf{u}^d = (u^d, v^d)$ describes the velocity and p^d the fluid pressure inside the porous medium. $\mathbb{K}$ is the hydraulic conductivity tensor. Here, only the case $\mathbb{K} = K\mathbb{I}$, $K > 0$ is considered. Sinks and sources are described by the force term f^d.

The free flow subproblem is modeled by using the Stokes equations for a viscous, incompressible, Newtonian fluid. The motion of the Stokes flow in the region Ω^f is described as

$$\begin{aligned} -\nabla \cdot \boldsymbol{\sigma}^f &= \mathbf{f}^f \quad \text{in } \Omega^f \ , \\ \nabla \cdot \mathbf{u}^f &= 0 \quad \text{in } \Omega^f \ , \end{aligned} \tag{2}$$

where $\mathbf{u}^f = (u^f, v^f)$ is the fluid velocity, $\mathbf{f}^f = (f_1^f, f_2^f)$ represents a prescribed force, and the fluid stress tensor $\boldsymbol{\sigma}^f := \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{pmatrix}$ is given by $\boldsymbol{\sigma}^f = -p^f \mathbf{I} + 2\nu \mathbf{D}(\mathbf{u}^f)$, with p^f denoting the fluid pressure, ν representing the fluid viscosity and where $\mathbf{D}(\mathbf{u}^f) = (\nabla \mathbf{u}^f + (\nabla \mathbf{u}^f)^T)/2$ is the strain tensor.

The Darcy and Stokes systems must be coupled across the internal interface Γ by adequate interface conditions. To describe such interface conditions, we fix the normal vector to the interface to be $\mathbf{n} = \mathbf{n}^f = -\mathbf{n}^d$ and we denote $\boldsymbol{\tau}$ as the tangential unit vector at the interface Γ. Across the interface Γ the continuity of fluxes and normal stresses must be imposed. This gives rise to the following two standard coupling conditions on Γ:

- Mass conservation:

$$\mathbf{u}^f \cdot \mathbf{n} = \mathbf{u}^d \cdot \mathbf{n} \quad \text{on } \Gamma \ . \tag{3}$$

- Balance of normal stresses (g the gravitational acceleration):

$$-\,\mathbf{n} \cdot \boldsymbol{\sigma}^f \cdot \mathbf{n} = g p^d \quad \text{on } \Gamma. \tag{4}$$

As third coupling condition, the so-called Beavers-Joseph-Saffman interface condition is widely used, which is supported by experimental findings. This condition relates the tangential velocity along the interface with the fluid stresses, that is,

$$\alpha \mathbf{u}^f \cdot \boldsymbol{\tau} + \boldsymbol{\tau} \cdot \boldsymbol{\sigma}^f \cdot \mathbf{n} = 0 \quad \text{on } \Gamma \ , \tag{5}$$

where α is a dimensionless parameter which needs to be experimentally determined and depends on the properties of the porous medium.

An alternative to this third interface condition neglects the second term in (5), giving rise to a no-slip interface condition,

$$\mathbf{u}^f \cdot \boldsymbol{\tau} = 0 \quad \text{on } \Gamma \ . \tag{6}$$

3. DISCRETIZATION

The finite volume method on a staggered grid is considered as the discretization scheme for the coupled Darcy/Stokes problem. By using this discretization we ensure that spurious oscillations do not appear in the numerical solution, and we obtain a mass conservative algorithm for the whole system. The computational domain is partitioned into square blocks of size $h \times h$, so that the grid is conforming at the interface Γ. For notational convenience, we choose equal-sized blocks but the description in the more general case would be straightforward. Different control volumes are defined depending on which variable is considered. In Figure 2, we represent in different colors the control volumes corresponding to the primary variables $u^{d/f}$, $v^{d/f}$ and $p^{d/f}$.

The discretizations for the mixed formulation of the Darcy problem and the Stokes equations have no particular difficulties. In this section, we mainly describe how we deal with the interface conditions. Our proposal is to obtain a special discrete equation for the unknowns at the internal interface, that is, for the vertical components of the velocity, see Figure 3. For this purpose, we integrate the momentum equation of the Stokes system over a half volume as displayed in red color in Figure 3, giving rise to the following equation

$$-\left(\frac{(\sigma_{xy})_e - (\sigma_{xy})_w}{h} + \frac{(\sigma_{yy})_n - (\sigma_{yy})_s}{h/2} \right) = (f_2^f)_{i,j+\frac{1}{2}}, \tag{7}$$

where, as can be seen in Figure 3, e and w denote locations at the interface, whereas n and s denote the locations of $p^f_{i,j+1}$ and $v^{d/f}_{i,j+\frac{1}{2}}$, respectively. The approximation of $(\sigma_{yy})_n$ is easily obtained as

$$(\sigma_{yy})_n = -p^f_{i,j+1} + \frac{2\nu}{h}(v^f_{i,j+\frac{3}{2}} - v^f_{i,j+\frac{1}{2}}), \tag{8}$$

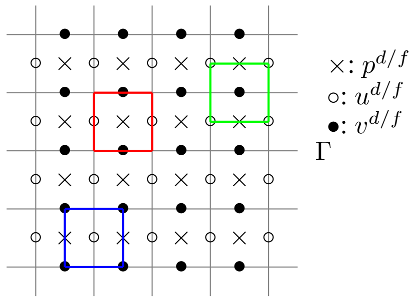


Figure 2. Staggered grid location of unknowns for the coupled model, and corresponding control volumes.

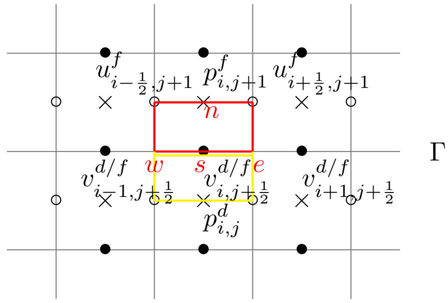


Figure 3. Staggered grid location of the unknowns for the interface conditions.

whereas the approximation of the other components of the stress tensor require more work by using the interface conditions. To approximate the component $(\sigma_{yy})_s$, we directly apply the interface condition (4), obtaining

$$(\sigma_{yy})_s = -gp^d_s. \tag{9}$$

The pressure p^d_s is not known at the interface, but it can be approximated by using the Darcy problem. By integrating the corresponding equation over a half volume as displayed in yellow in Figure 3, we obtain,

$$K^{-1}v^d_{i,j+\frac{1}{2}} + \frac{p^d_s - p^d_{i,j}}{h/2} = 0. \tag{10}$$

Using this equation in (9), the approximation reads,

$$(\sigma_{yy})_s = -gp^d_{i,j} + \frac{gh}{2K}v^d_{i,j+\frac{1}{2}}. \tag{11}$$

To approximate the remaining components of the stress tensor, we need to use the no-slip or the Beavers-Joseph-Saffman interface condition. Here, we consider the latter since it is the most involved case. The standard approximation of the Beavers-Joseph-Saffman condition (5) at the location denoted by e reads

$$\alpha u^f_e - \nu\left(\frac{u^f_{i+\frac{1}{2},j+1} - u^f_e}{h/2} + \frac{v^f_{i+1,j+\frac{1}{2}} - v^f_{i,j+\frac{1}{2}}}{h}\right) = 0\,. \tag{12}$$

Here, u^f_e can be obtained from (12) and substituted into the standard approximation of the stress $(\sigma_{xy})_e$, resulting in

$$(\sigma_{xy})_e = \nu\left(\frac{u^f_{i+\frac{1}{2},j+1} - u^f_e}{h/2} + \frac{v^f_{i+1,j+\frac{1}{2}} - v^f_{i,j+\frac{1}{2}}}{h}\right) = \frac{2\nu m}{h}u^f_{i+\frac{1}{2},j+1} + \nu m\frac{v^f_{i+1,j+\frac{1}{2}} - v^f_{i,j+\frac{1}{2}}}{h}, \tag{13}$$

where $m = \left(1 - \frac{2\nu}{h\alpha+2\nu}\right)$. The approximation of $(\sigma_{xy})_w$ can be calculated in a similar way. The discrete equation for the vertical velocities for the Stokes problem at the interface is thus obtained by substituting (8), (11) and (13) into equation (7), giving

$$\begin{aligned}\frac{2\nu m}{h^2}u^f_{i-\frac{1}{2},j+1} - \frac{2\nu m}{h^2}u^f_{i+\frac{1}{2},j+1} - \frac{\nu m}{h^2}v^f_{i+1,j+\frac{1}{2}} - \frac{\nu m}{h^2}v^f_{i-1,j+\frac{1}{2}} - \frac{4\nu}{h^2}v^f_{i,j+\frac{3}{2}}\\ +(\frac{2\nu m}{h^2} + \frac{4\nu}{h^2} + \frac{g}{K})v^f_{i,j+\frac{1}{2}} + \frac{2}{h}p^f_{i,j+1} - \frac{2g}{h}p^d_{i,j} = (f^f_2)_{i,j+\frac{1}{2}},\end{aligned} \tag{14}$$

where we have used the interface condition $v^d_{i,j+\frac{1}{2}} = v^f_{i,j+\frac{1}{2}}$.

4. NUMERICAL METHOD

This section is devoted to the design of a monolithic geometric multigrid for the coupled Stokes/Darcy problem. For this purpose, we will study the application of multigrid methods based on Uzawa smoothers to the Darcy and Stokes problems separately. In this analysis we will take into account the development of an LFA technique to obtain suitable parameters for these methods. These algorithms will form the basis to construct a monolithic multigrid for the coupled problem. This will be possible since the individual Stokes and Darcy systems, as well as the fully coupled problem, lead to saddle point linear systems of the form $\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix}\begin{pmatrix} \mathbf{u} \\ p \end{pmatrix} = \begin{pmatrix} \mathbf{g} \\ f \end{pmatrix}$, by choosing an adequate arrangement of the unknowns. For both problems B^T and B represent the discrete gradient and the minus discrete divergence operators, respectively, and A is the discrete representation of the Laplace-type operator $-\nu\Delta$ for the Stokes equations, or $K^{-1}I$ for the Darcy equation. For the coupled problem, rearranging the vector of unknowns to order first the velocities for both problems and thereafter the pressure unknonws, we obtain the following linear system,

$$\begin{pmatrix} A^d & 0 & (B^d)^T & 0 \\ 0 & A^f & 0 & (B^f)^T \\ B^d & 0 & 0 & 0 \\ 0 & B^f & 0 & 0 \end{pmatrix}\begin{pmatrix} \mathbf{u}^d \\ \mathbf{u}^f \\ p^d \\ p^f \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{f}^f \\ f^d \\ 0 \end{pmatrix}, \tag{15}$$

where the system matrix in (15) has the saddle point structure as well. Due to this structure of the coupled problem, a geometric multigrid method together with an Uzawa smoother can be applied for the whole system. Regarding the multigrid, geometric grid coarsening is chosen here, as we will deal with regular Cartesian grids. The sequence of coarse grids is obtained by doubling the mesh size in each spatial direction. As we will see, the choice of adequate relaxation parameters for the Uzawa smoother on each subproblem will be crucial for excellent multigrid convergence.

Uzawa smoother. The Uzawa smoother is obtained by splitting the discrete operator. Therefore, from a given approximation of the solution to the system $(\mathbf{u}, p)^T$, the relaxed approximation $(\widehat{\mathbf{u}}, \widehat{p})^T$ is computed according to the decoupled Uzawa smoother in the following way

$$\begin{pmatrix} M_A & 0 \\ B & -\omega^{-1} I \end{pmatrix}\begin{pmatrix} \widehat{\mathbf{u}} \\ \widehat{p} \end{pmatrix} = \begin{pmatrix} M_A - A & -B^T \\ 0 & -\omega^{-1} I \end{pmatrix}\begin{pmatrix} \mathbf{u} \\ p \end{pmatrix} + \begin{pmatrix} \mathbf{g} \\ f \end{pmatrix}. \tag{16}$$

where M_A is a typical smoother for A and ω is some positive parameter. M_A makes the approach less costly because of the inexact solve for velocities at each iteration. The symmetric Gauss-Seidel method consists of one forward and one backward sweep for all velocities in the computational domain. Numerical experiments in [14] revealed that, for essentially the same cost, the convergence associated with the symmetric Gauss-Seidel operator M_A is most efficient. So, this variant is the one that we extend to the Darcy equation.

4.1. *Local Fourier analysis*

Basis of LFA. To perform LFA, all discrete operators are assumed to be defined on an infinite grid G_h, and boundary conditions are neglected. The basic idea of LFA is that all occurring multigrid components, the discrete approximation and its corresponding error or residual can be represented by formal linear combinations of Fourier modes $\varphi_h(\boldsymbol{\theta}, \boldsymbol{x})$ (see [14]), which form a unitary basis of the space of infinite grid functions. Here $\boldsymbol{\theta} \in \boldsymbol{\Theta} := (-\pi, \pi]^2$ and $\boldsymbol{x}$ denotes the nodes location. For the analysis, we distinguish high and low frequency components on G_h as $\boldsymbol{\Theta}^{2h}_{low} := (-\frac{\pi}{2}, \frac{\pi}{2}]^2$ and $\boldsymbol{\Theta}^{2h}_{high} := \boldsymbol{\Theta} \backslash \boldsymbol{\Theta}^{2h}_{low}$. To study how efficiently high frequency error components are eliminated, smoothing factor μ is defined as: $\mu := \sup_{\boldsymbol{\theta} \in \boldsymbol{\Theta}^{2h}_{high}} \rho(S_h(\boldsymbol{\theta}))$, where $S_h(\boldsymbol{\theta})$ represents the Fourier symbol of the relaxation operator. In particular, the iteration operator of the two-grid method is given by $M_{h,2h}$. Since the representation of $M_{h,2h}$ on the Fourier space has a block-diagonal structure, it is possible to efficiently calculate the LFA two-grid convergence factor as $\rho = \rho(M_{h,2h})$.

LFA for the Uzawa smoother A detailed study of the Uzawa smoother in the framework of LFA was already done in [14]. An analytic bound of the smoothing factor of the Uzawa smoother was given for a family of Stokes problems, showing a satisfactory approximation of the exact smoothing factor. In that work, it was proved that

$\mu \leq \bar{\mu} = \max\left((\mu_A)^{1/2}, \mu_S\right)$, where μ_A is the smoothing factor of M_A and μ_S can be interpreted as the smoothing factor of the Richardson iteration for the Schur complement, i.e., $\mu_S := \sup_{\Theta^{2h}_{high}} \rho\left(I - \omega\left(BA^{-1}B^T\right)\right)$. There are no particular difficulties to obtain bounds for μ_A, since LFA results for many scalar elliptic PDEs are available in the literature, see for example [29]. However, to estimate μ_S is somewhat involved since information about the eigenvalues of the Schur complement is needed. In particular, the bound of μ_S is determined by the maximum and minimum eigenvalues on the high frequencies, that is,

$$\max_{\boldsymbol{\theta}\in\Theta^{2h}_{high}} \left(\widetilde{B}(\boldsymbol{\theta})\widetilde{A^{-1}}(\boldsymbol{\theta})\widetilde{B^T}(\boldsymbol{\theta})\right) \leq \beta_{\max}, \quad \min_{\boldsymbol{\theta}\in\Theta^{2h}_{high}} \left(\widetilde{B}(\boldsymbol{\theta})\widetilde{A^{-1}}(\boldsymbol{\theta})\widetilde{B^T}(\boldsymbol{\theta})\right) \geq \beta_{\min}, \tag{17}$$

with $\widetilde{B}(\boldsymbol{\theta})$, $\widetilde{A^{-1}}(\boldsymbol{\theta})$ and $\widetilde{B^T}(\boldsymbol{\theta})$ the symbols or Fourier representations of operators B, A^{-1} and B^T for a fixed frequency $\boldsymbol{\theta}$. We define $\kappa_\beta = \frac{\beta_{\max}}{\beta_{\min}}$, and by choosing a positive real number τ such that $\tau < 2$ (to ensure that $\mu_S < 1$), the bound for μ_S is obtained as $\mu_S \leq \max\left(\tau - 1, 1 - \frac{\tau}{\kappa_\beta}\right)$. By choosing a value of τ to minimize the expression of μ_S, we obtain an optimal relaxation parameter for the Uzawa smoother as $\omega = \frac{\tau}{\beta_{\max}}$. Next, we apply this analysis to obtain approximations of the smoothing factor of the Uzawa smoother for our problem, as well as optimal relaxation parameters for the Richardson iteration involved in the relaxation process.

In [14], the following bound for the smoothing factor of the Uzawa smoother was obtained in the case of Stokes equations, $\bar{\mu} = \max(0.5, \tau - 1)$, by choosing the optimal relaxation parameter $\omega = \tau\nu$. Notice that $\mu_A = 0.25$ for the symmetric Gauss-Seidel for the Laplace operator, and therefore $(\mu_A)^{1/2} = 0.5$. These results can be directly used for our free flow problem.

Uzawa smoother analysis for Darcy equation. We work out the analysis for Darcy's equation in order to obtain a suitable parameter ω for the part corresponding to the Richardson iteration for the pressure, as well as an approximation of the smoothing factor of the Uzawa smoother.

Following the general analysis in the previous section to obtain $\beta_{\max}$ and $\beta_{\min}$, we will make use of the equality $\widetilde{B}(\boldsymbol{\theta})\widetilde{A^{-1}}(\boldsymbol{\theta})\widetilde{B^T}(\boldsymbol{\theta}) = K\widetilde{B}(\boldsymbol{\theta})\widetilde{B^T}(\boldsymbol{\theta}) = -K\widetilde{\Delta}(\boldsymbol{\theta})$. From this result, it is straightforward to obtain $\beta_{max} = \dfrac{8K}{h^2}$ and $\beta_{min} = \dfrac{2K}{h^2}$, which implies $\kappa_\beta = \dfrac{\beta_{\max}}{\beta_{\min}} = 4$. Choosing $\tau = 1.6$, which gives the lowest value of $\max\left(\tau - 1, 1 - \frac{\tau}{\kappa_\beta}\right)$, the smoothing factor is bounded by 0.6, independently of the value of K. This theoretical bound for the smoothing factor $\bar{\mu}$ matches perfectly with the value μ predicted by the local Fourier analysis. Moreover, the relaxation parameter is given by the expression $\omega = \dfrac{h^2}{5K}$. Parameter ω depends on the grid size, and therefore it will be different on each grid of the hierarchy used in the multigrid method.

4.2. *Multigrid for the coupled Darcy/Stokes problem*

Due to the saddle point structure of the coupled problem, a geometric multigrid method together with an Uzawa smoother, can be applied for the whole system. For this purpose, in the smoothing process, all velocity unknowns are relaxed before the pressure unknowns will be updated. The relaxation parameter ω for the Richardson iteration for the Schur complement has to be chosen differently if we are updating pressure unknowns from the Darcy or the Stokes problems. For the rest of the components, the same operators can be used at every grid point since the discretization for both problems is performed with the same staggered arrangement of unknowns.

The proposed multigrid method for the coupled Darcy/Stokes problem can also be implemented as a multiblock version in which the Darcy and Stokes domains are assumed to be two different blocks. This is appealing from a practical point of view, for example when one has to solve the coupled problem by using two different codes. Moreover, this multiblock approach is easily parallelizable. Next, we describe in detail how this implementation can be done.

Multiblock multigrid algorithm. We divide our domain into two different blocks corresponding to the Darcy and Stokes domains. In this way, the original staggered grid is split into two different sub-grids. Since in this version of the algorithm it is necessary to transfer information between both blocks, the mesh corresponding to the Stokes domain is extended by adding an overlap region of one cell length, as can be seen in Figure 4. Next, we explain in

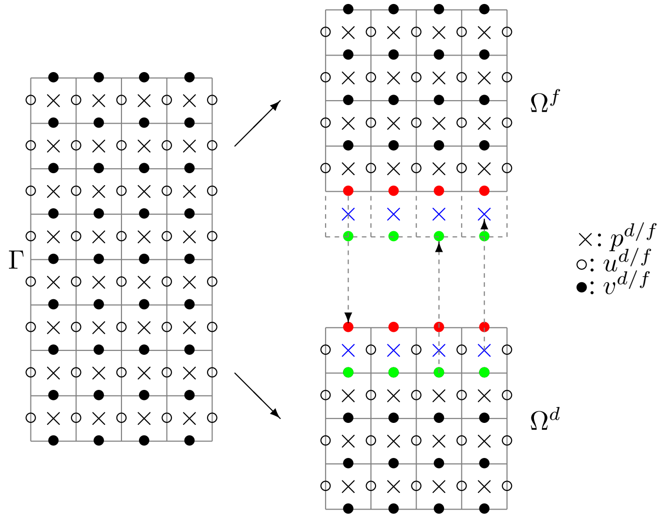


Figure 4. Communications between two partitioned subgrids.

detail the two-grid version of the multiblock algorithm. For simplicity in the presentation of the algorithm, we use pre-smoothing but no post-smoothing. By recursion, the multigrid version follows straightforwardly.

Multiblock two-grid algorithm: (with pre-smoothing but no post-smoothing)

1. Relax velocity unknowns for both blocks.
2. *Stokes to Darcy transfer*: vertical Stokes velocity unknowns at the interface are transferred to the Darcy block (see the red dots in Figure 4).
3. Update pressure unknowns by the Richardson iteration with the optimal relaxation parameters corresponding to each block.
4. *Darcy to Stokes transfer*: Darcy pressure unknowns are transferred to the Stokes overlap region (see the blue crosses in Figure 4).
5. Compute the residual.
6. *Darcy to Stokes transfer*: the residual of the vertical Darcy velocity unknowns is transferred to the Stokes overlap region (see the green dots in Figure 4).
7. Restrict the residual.
8. Solve exactly the defect equation on the coarsest grid.
9. *Stokes to Darcy transfer*: vertical Stokes velocity unknowns at the interface are transferred to the Darcy block.
10. Interpolate the error and correct the approximation to the solution.

This multiblock algorithm requires only little data communication. In particular, each communication step involves transfer of information in only one way. Moreover, each stage in the algorithm can be performed in parallel since the data required for each operation is available in the same process. Finally, although this multiblock approach can be cast into the class of domain decomposition (DD) methods, we wish to emphasize that in our case the communication between both Darcy and Stokes problems is performed *on each level in the hierarchy* instead of only on the finest grid as usual in the DD methods. This is crucial to achieve a highly efficient solver for this coupled problem, as we will see in the numerical experiments section.

Local Fourier analysis results. In this section, we confirm that the asymptotic convergence factor of the monolithic multigrid based on the Uzawa smoother for the coupled problem can be estimated with a high accuracy by means of the worst of the two-grid convergence factors predicted by LFA for the individual Darcy and Stokes subproblems. In Table I, we display the two-grid convergence factors predicted by the LFA for the Darcy problem varying the hydraulic conductivity K, and for the Stokes equations for different values of the viscosity ν. These results are obtained for different numbers of smoothing steps, $\nu_1 + \nu_2$. From this table, we can observe the robustness of the multigrid method based on Uzawa smoother for each subproblem, separately. In Table II, we show the asymptotic

	Darcy			Stokes		
$\nu_1 + \nu_2$	$K = 1$	$K = 10^{-3}$	$K = 10^{-6}$	$\nu = 1$	$\nu = 10^{-3}$	$\nu = 10^{-6}$
2	0.600	0.600	0.600	0.304	0.304	0.304
3	0.360	0.360	0.360	0.143	0.143	0.143
4	0.216	0.216	0.216	0.081	0.081	0.081

Table I. Two-grid convergence factors, ρ predicted by LFA for Darcy and Stokes subproblems, separately, for different values of the parameters K and ν and different numbers of smoothing steps, $\nu_1 + \nu_2$.

convergence factors experimentally obtained by using the monolithic multigrid method based on Uzawa smoother for the Darcy/Stokes coupled problem. These values have been computed on a fine-grid of size $h = 1/128$, and by using a random initial guess and zero right-hand side in order to avoid round-off errors. Comparing Tables I and II, we observe that these factors match perfectly with the worst of the two-grid convergence factors predicted by LFA for both separate subproblems. This means that the treatment of the discretization at the interface as well as the implementation of the Uzawa smoother for the whole coupled problem have been performed in the most efficient way.

	K	1			10^{-3}			10^{-6}		
	ν	1	10^{-3}	10^{-6}	1	10^{-3}	10^{-6}	1	10^{-3}	10^{-6}
$\nu_1 + \nu_2$	2	0.60	0.60	0.60	0.60	0.60	0.60	0.60	0.60	0.60
	3	0.36	0.36	0.36	0.36	0.36	0.36	0.36	0.36	0.36
	4	0.22	0.22	0.22	0.22	0.22	0.22	0.21	0.22	0.21

Table II. Asymptotic convergence factors, ρ_h, for the multigrid based on Uzawa smoother for the coupled Darcy/Stokes problem, for different values of the physical parameters K and ν and different numbers of smoothing steps $\nu_1 + \nu_2$.

Remark. Due to the fact that the Uzawa smoother for the coupled problem is performed as a relaxation for the whole domain, it is not possible to reduce the number of smoothing steps for one of the subproblems, which would be very appealing in order to balance the computational work needed to smooth the two different problems. This strategy can be used if the smoother for the coupled problem is implemented as an Uzawa for the porous medium subdomain combined with an Uzawa for the free flow region, instead of the Uzawa in the whole domain that we propose here. However, although this strategy results in a good balance of the computational cost, it gives rise to a deterioration of the convergence of the multigrid algorithm for the coupled problem when small values of the physical parameters are considered, and it provides very good results only when these parameters remain big enough.

5. NUMERICAL EXPERIMENT

We present a numerical test in order to study the accuracy of the discrete scheme and the convergence and robustness of the proposed multigrid method based on the Uzawa smoother with respect to different values of the kinematic viscosity ν and the hydraulic conductivity K. For the implementation, we will consider the optimal relaxation parameters for the Richardson iteration defined in Section 4, with values of $\tau = 1$ for Stokes and $\tau = 1.6$ for the Darcy problem. For Stokes it follows that $\omega = \nu$, that is, the relaxation parameter is fixed on all grids and equal to the viscosity of the fluid; and $\omega = \dfrac{h^2}{5K}$ in the Darcy domain, so ω depends on K which is the hydraulic conductivity of the porous media and on the size of the grid (different on each mesh in the hierarchy).

In the numerical experiment, the initial solution is chosen to be zero, and the stopping criterion is to reduce the maximum initial residual by a factor of 10^{-10} in maximum norm. Moreover, for simplicity we consider uniform meshes with grid-size h in both directions on each subdomain.

We consider a more complicated and realistic numerical test in which the Beavers-Joseph-Saffman interface condition is prescribed. In this case, the domain $\Omega = (0, 1) \times (-1, 1)$ is divided into a porous medium part $\Omega^d =$

$(0,1)\times(-1,0)$ and a free-flow subdomain $\Omega^f = (0,1)\times(0,1)$ by the interface $\Gamma = (0,1)\times\{0\}$. The source terms and the boundary conditions are chosen such that the analytic solution of the coupled Darcy/Stokes problem is as follows,

$$\begin{aligned} \mathbf{u}^d(x,y) &= \begin{pmatrix} u^d(x,y) \\ v^d(x,y) \end{pmatrix} = \begin{pmatrix} -Ke^y\cos x \\ -Ke^y\sin x \end{pmatrix}, \quad p^d(x,y) = e^y\sin x, \\ \mathbf{u}^f(x,y) &= \begin{pmatrix} u^f(x,y) \\ v^f(x,y) \end{pmatrix} = \begin{pmatrix} \lambda'(y)\cos x \\ \lambda(y)\cos x \end{pmatrix}, \quad p^f(x,y) = 0, \end{aligned} \tag{18}$$

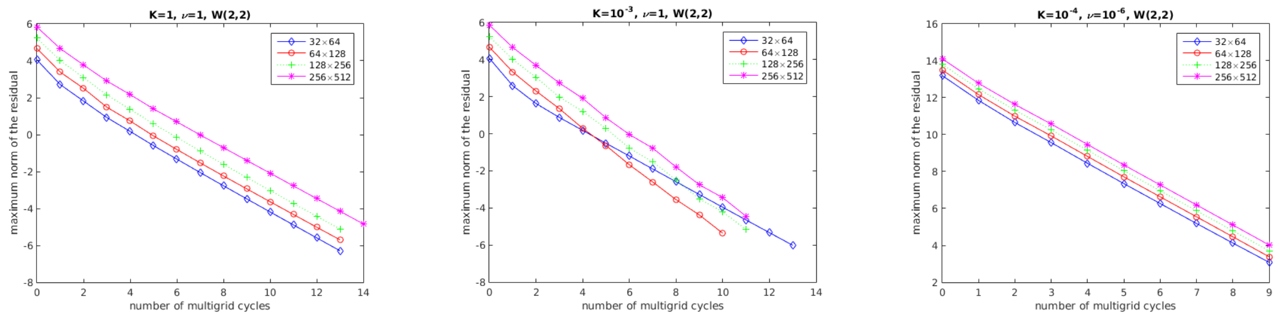
where $\lambda(y) = -K - \dfrac{g\,y}{2\nu} + (-\dfrac{g}{4\nu^2} + \dfrac{K}{2})y^2$. At the outer boundaries of the free-flow domain, Dirichlet boundary conditions for velocities are prescribed. In the case of the porous medium, the pressure is fixed at the bottom $(0,1)\times\{-1\}$, whereas Dirichlet conditions for velocities are imposed at the lateral walls. Along the internal interface Γ, the Beavers-Joseph-Saffman condition (5) is taken into account.

We begin by comparing the numerical solution with the given exact solution for fixed values of the parameters $\nu = K = 1$ and for different grid-sizes $h = 1/2^k$ for $k = 5,6,7,8$. In Table III we display the maximum norm of the error obtained for each variable, and it can be seen that second order accuracy is obtained for all variables except for the pressure in the free-flow subdomain where we achieve first order accuracy.

	32×64	64×128	128×256	256×512
u^d	5.50×10^{-5}	1.42×10^{-5}	3.63×10^{-6}	9.19×10^{-7}
v^d	1.47×10^{-4}	4.09×10^{-5}	1.19×10^{-5}	3.38×10^{-6}
p^d	3.53×10^{-5}	9.11×10^{-6}	2.32×10^{-6}	5.84×10^{-7}
u^f	4.68×10^{-5}	1.21×10^{-5}	3.06×10^{-6}	7.71×10^{-7}
v^f	1.13×10^{-4}	2.97×10^{-5}	7.66×10^{-6}	1.95×10^{-6}
p^f	9.38×10^{-3}	4.74×10^{-3}	2.38×10^{-3}	1.19×10^{-3}

Table III. Maximum norm errors of variables $u^{d/f}$, $v^{d/f}$ $p^{d/f}$ for different grid-sizes, by considering fixed values $\nu = 1$ and $K = 1$, and prescribing the Beavers-Joseph-Saffman condition at the interface with $\alpha = 1$.

Regarding the performance of the monolithic multigrid method for the coupled problem considered in this numerical test, we display in Figure 5 (a) the history of the convergence of the algorithm by using a $W(2,2)-$cycle for different grids and $\nu = K = 1$. It is clear that the convergence is independent of the mesh size and that the method performs efficiently since only needs around 13 iterations to achieve the required stopping criterion. In Figure 5 (b) and (c), the robustness of the proposed multigrid method is displayed, since for different values of ν and K and different grid-sizes the convergence of the algorithm is highly satisfactory and independent of the parameters. We can



(a) (b) (c)

Figure 5. History of the convergence of the $W(2,2)-$multigrid method when the Beavers-Joseph-Saffman interface condition is considered for different values of the physical parameters:
(a) $\nu = 1$, $K = 1$, (b) $\nu = 10^{-3}$, $K = 1$, and (c) $\nu = 10^{-6}$, $K = 10^{-4}$.

observe that with the more complicated Beavers-Joseph-Saffman condition at the interface Γ, the results provided by the proposed multigrid method for the coupled Darcy/Stokes problem are highly satisfactory.

6. CONCLUSIONS

In this paper, we investigated the multigrid convergence of a coupled system consisting of a porous medium and incompressible flow. For this purpose, a coupled model based on the Darcy equation and the incompressible Stokes equations with appropriate internal interface conditions is formulated. The model is discretized by finite volumes on a staggered grid, and special care has been taken regarding the accurate discretization at the interface. We focused on an efficient multigrid algorithm with a decoupled Uzawa smoother for the coupled problem. By Local Fourier Analysis we have selected suitable relaxation parameters for both systems, and we have confirmed the global convergence of the monolithic multigrid which results to be the worst of the convergence factors between both the individual Darcy and Stokes subproblems. Numerical tests have shown a highly satisfactory convergence of our multigrid method for the coupled system. The algorithm performed very well in numerical experiments for a wide range of physical parameter values.

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