Indefinite preconditioning of the Coupled Stokes-Darcy System

Prince Chidyagwai* Scott Ladenheim† Daniel B. Szyld‡

Abstract

We propose the use of an indefinite (constraint) preconditioner for the iterative solution of the linear system arising from the finite element discretization of coupled Stokes-Darcy flow. We provide spectral bounds for the preconditioned system which are independent of the underlying mesh size. Numerical experiments show the effectiveness of our approach.

1 Introduction

The coupled Stokes-Darcy model is an important mathematical description of two different flows in a specified domain. In one subregion of the domain, a freely flowing fluid is described by the Stokes equations and in the other region, the flow is governed by Darcy's law. The equations are coupled together by conditions on the interface. Physically, the model can represent how fluids percolate through a porous medium but can also represent certain filtration processes, such as the filtration of blood through arterial vessel walls.

We solve this system of partial differential equations numerically using the finite element method. The Stokes domain is discretized using continuous finite element spaces that satisfy an inf-sup condition. In the Darcy domain, we consider both continuous functions (standard finite elements) and discontinuous polynomials (discontinuous Galerkin methods). In both cases, the discretization leads to a system of equations that is sparse, non-symmetric (and non-symmetrizable in the DG case) and of saddle point form.

The solution of this resulting linear system is our focus, in particular the choice of preconditioner. We propose solving the system of equations using preconditioned GMRES with an indefinite or constraint preconditioner. This preconditioner mimics the structure of the original system matrix. We prove that the convergence of GMRES using this preconditioner is bounded independently of the mesh discretization. Our numerical results show that the indefinite preconditioner outperforms other standard block diagonal and block lower triangular preconditioners both with respect to iteration count and CPU times.

The paper is structured as follows, in section 2 we introduce the governing equations of the coupled Stokes-Darcy model along with the corresponding weak form to derive the linear system of interest to be solved. Section 3 introduces the preconditioners to be considered in addition to proving that the use of an indefinite preconditioner leads to residual convergence rates that are independent of the discretization parameter. Section 4 contains the numerical experiments demonstrating the effectiveness of our approach. Section 5 contains the concluding remarks.

*Department of Mathematics and Statistics, Loyola University Maryland, 4501 N. Charles Street, Baltimore, MD 21210. (pchidyagwai@loyola.edu)

†Department of Mathematics, Temple University, 1805 N Broad Street, Philadelphia, PA 19122. (saladenh@temple.edu)

‡Department of Mathematics, Temple University, 1805 N Broad Street, Philadelphia, PA 19122. (szyld@temple.edu)

2 The coupled Stokes-Darcy Model

We consider fluid flowing in a domain $\Omega = \Omega_1 \cup \Omega_2$; see Figure 1. For simplicity, in this paper we consider the case where Ω is a rectangle, but most of what we say equally applies to other geometries.The flow in Ω_1 is modeled by the Stokes equations

$$-\nabla \cdot (2\nu D(\mathbf{u}_1) - p_1 I) = \mathbf{f}_1, \quad \text{in } \Omega_1, \tag{2.1a}$$

$$\nabla \cdot \mathbf{u}_1 = 0, \quad \text{in } \Omega_1, \tag{2.1b}$$

$$\mathbf{u}_1 = 0, \quad \text{on } \Gamma_1 := \partial\Omega_1 \setminus \Gamma_{12}. \tag{2.1c}$$

The velocity and pressure in Ω_1 are denoted by $\mathbf{u}_1$, p_1, respectively. The coefficient $\nu > 0$ is the kinematic viscosity, the function $\mathbf{f}_1$ is an external force acting on the fluid, $\mathbf{I}$ is the identity matrix and $D(\mathbf{u}_1) = \frac{1}{2}(\nabla\mathbf{u}_1 + \nabla\mathbf{u}_1^T)$ is the deformation matrix. Lastly, Γ_1 is the boundary of the domain Ω_1 excluding the interface Γ_{12}.

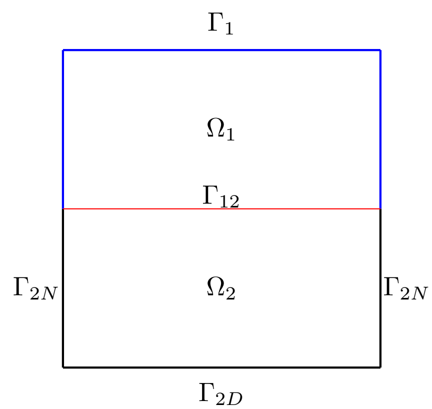


Figure 1: The domain $\Omega = \Omega_1 \cup \Omega_2$. The Stokes flow region is Ω_1. The Darcy region is Ω_2. The stokes boundary (excluding the interface), Γ_1, is colored blue. The interface Γ_{12} is colored red. The Darcy boundary is composed of Dirichlet (Γ_{2D}) and Neumann (Γ_{2N}) parts.

The flow in Ω_2 is modeled by Darcy's Law

$$-\nabla \cdot \mathbf{K}\nabla p_2 = f_2, \quad \text{in } \Omega_2, \tag{2.2a}$$

$$-\mathbf{K}\nabla p_2 = \mathbf{u}_2, \quad \text{in } \Omega_2, \tag{2.2b}$$

$$p_2 = g_D, \quad \text{on } \Gamma_{2D}, \tag{2.2c}$$

$$\mathbf{K}\nabla p_2 \cdot \mathbf{n}_2 = g_N, \quad \text{on } \Gamma_{2N}. \tag{2.2d}$$

The velocity and pressure in Ω_2 are denoted $\mathbf{u}_2$, p_2, respectively and the function f_2 is an external force acting on the fluid. The functions g_D and g_N are prescribed on the portions of the boundary corresponding to the Dirichlet (Γ_{2D}) and Neumann (Γ_{2N}) conditions respectively, so that $\Gamma_2 := \partial\Omega 2 \backslash \Gamma_{12} = \Gamma_{2D} \cup \Gamma_{2N}$. The symmetric positive definite matrix $\mathbf{K}$ represents the hydraulic conductivity of the fluid and the vector $\mathbf{n}_2$ denotes the outward unit normal vector to Γ_{2N}. For isotropic flow we will have that the hydraulic conductivity matrix is a scaled identity matrix with scaling factor κ, i.e., $\mathbf{K} = \kappa\mathbf{I}$.

Let $\mathbf{n}_{12}, \tau_{12}$, denote the unit normal vector directed from Ω_1 to Ω_2 and unit tangential vector to the interface, respectively. The model is completed by specifying the following coupling (interface) conditions between the two domains

$$\mathbf{u}_1 \cdot n_{12} = -\mathbf{K}\nabla p_2 \cdot n_{12}, \tag{2.3a}$$
$$(-2\nu D(\mathbf{u}_1)\mathbf{n}_{12} + p_1\mathbf{n}_{12}) \cdot \mathbf{n}_{12} = p_2, \tag{2.3b}$$
$$\mathbf{u}_1 \cdot \tau_{12} = -2\nu G(D(\mathbf{u}_1)\mathbf{n}_{12}) \cdot \tau_{12}, \tag{2.3c}$$

where (2.3a) ensures mass conservation across the interface, (2.3b) ensures the balance of of normal forces across the interface, and (2.3c) is the Beavers-Joseph-Saffman (BJS) law, with the BJS constant G that is determined experimentally [1, 9].

To introduce the weak form of these equations, let

$$X_1 = \{\mathbf{v}_1 \in (H^1(\Omega_1))^2 : \ \mathbf{v}_1 = 0 \text{ on } \Gamma_1\}, \quad Q_1 = L^2(\Omega_1)$$

be the Stokes velocity and pressure spaces respectively and let

$$Q_2 = \{q_2 \in H^1(\Omega_2) : \ q_2 = 0 \text{ on } \Gamma_{2D}\}.$$

The weak formulation of the coupled Stokes/Darcy problem (2.1), (2.2), (2.3), is to find $\mathbf{u}_1 \in X_1$, $p_1 \in Q_1$, $p_2 \in Q_2$ such that

$$a(\mathbf{u}_1, p_2; \mathbf{v}_1, q_2) + b^*(p_1, \mathbf{v}_1) = \mathbf{f}(\mathbf{v}_1, q_2) \quad \forall \mathbf{v}_1 \in X_1, \ \forall q_2 \in Q_2, \tag{2.4a}$$
$$b(\mathbf{u}_1, q_1) = 0 \quad \forall q_1 \in Q_1, \tag{2.4b}$$

where

$$a(\mathbf{u}_1, p_2; \mathbf{v}_1, q_2) = a_{\Omega_1}(\mathbf{u}_1, \mathbf{v}_1) + a_{\Omega_2}(p_2, q_2) + a_{\Gamma_{12}}(\mathbf{u}_1, p_2; \mathbf{v}_1, q_2),$$
$$b(\mathbf{u}_1, q_1) = -\int_{\Omega_1} (\nabla \cdot \mathbf{u}_1) q_1 \, dx,$$

and

$$a_{\Omega_1}(\mathbf{u}_1, \mathbf{v}_1) = 2\nu \int_{\Omega_1} D(\mathbf{u}_1) : D(\mathbf{v}_1) + \frac{1}{G}\int_{\Gamma_{12}} (\mathbf{u}_1 \cdot \tau_{12})(\mathbf{v}_1 \cdot \tau_{12}),$$
$$a_{\Omega_2}(p_2, q_2) = \int_{\Omega_2} \mathbf{K}\nabla p_2 \cdot \nabla q_2,$$
$$a_{\Gamma_{12}}(\mathbf{u}_1, p_2; \mathbf{v}_1, q_2) = \int_{\Gamma_{12}} (p_2\mathbf{v}_1 - q_2\mathbf{u}_1) \cdot \mathbf{n}_{12}.$$

Lastly,

$$\mathbf{f}(\mathbf{v}_1, q_2) = \int_{\Omega_1} \mathbf{f}_1 \cdot \mathbf{v}_1 + \int_{\Omega_2} f_2 q_2 + \int_{\Gamma_{2N}} g_N q_2. \tag{2.6}$$

The next step is to select appropriate finite element spaces, that is, choose $X_1^h \subset X_1$, $Q_1^h \subset Q_1$ satisfying a discrete inf-sup condition for the Stokes velocity and pressure in addition to selecting $Q_2^h \subset Q_2$ for the finite Darcy pressure space.

The discrete version of (2.4) is the following system of equations:

$$\mathcal{A}x = \begin{bmatrix} A_{\Omega_2} & A_{\Gamma_{12}}^T & 0 \\ -A_{\Gamma_{12}} & A_{\Omega_1} & B^T \\ 0 & B & 0 \end{bmatrix} \begin{bmatrix} p_{\Omega_2} \\ u_{\Omega_1} \\ p_{\Omega_1} \end{bmatrix} = \begin{bmatrix} f_h \\ \mathbf{f}_h \\ g_h \end{bmatrix} = b. \tag{2.7}$$

Setting

$$A = \begin{bmatrix} A_{\Omega_2} & A_{\Gamma_{12}}^T \\ -A_{\Gamma_{12}} & A_{\Omega_1} \end{bmatrix}, \quad C = \begin{bmatrix} 0 & B \end{bmatrix}$$

it is easy to see that the system of equations (2.7) is of saddle point form (see e.g. [2])

$$\mathcal{A} = \begin{bmatrix} A & C^T \\ C & 0 \end{bmatrix}.$$

We remark that if the continuous Galerkin method is used to solve for the flow in both the Stokes and Darcy regions, then the matrices A_{Ω_1}, A_{Ω_2} will both be symmetric and positive definite. However, due to the interface block $A_{\Gamma_{12}}$, the $(1,1)$ block, A, of the saddle point matrix $\mathcal{A}$ is non-symmetric. However one can scale the second row of the block matrix $\mathcal{A}$ in (2.7) to obtain a symmetric but now indefinite matrix A. Additionally, if one models the flow in the Darcy region with the discontinuous Galerkin method, as proposed in [4], the matrix A_{Ω_2} is no longer symmetric.

3 Preconditioning

To determine the discrete Stokes velocity, Stokes pressure and Darcy pressure we solve the large, sparse and non-symmetric saddle point matrix $\mathcal{A}$ with preconditioned GMRES [8]. We consider the following preconditioners

$$P_+ = \begin{bmatrix} A_{\Omega_2} & 0 & 0 \\ 0 & A_{\Omega_1} & 0 \\ 0 & 0 & I \end{bmatrix}, \qquad P_{T_1}(\rho) = \begin{bmatrix} A_{\Omega_2} & 0 & 0 \\ 0 & A_{\Omega_1} & 0 \\ 0 & B & -\rho I \end{bmatrix},$$

$$P_{T_2}(\rho) = \begin{bmatrix} A_{\Omega_2} & 0 & 0 \\ -A_{\Gamma_{12}} & A_{\Omega_1} & 0 \\ 0 & B & -\rho I \end{bmatrix}, \quad P_C(\rho) = \begin{bmatrix} A_{\Omega_2} & A_{\Gamma_{12}}^T & 0 \\ -A_{\Gamma_{12}} & A_{\Omega_1} & 0 \\ 0 & B & -\rho I \end{bmatrix}.$$

These preconditioners were examined in [3] and are standard block diagonal and block triangular preconditioners. Moreover, utilizing the theory established in [6], Cai, Mu and Xu [3] showed that the preconditioned operator $P^{-1}\mathcal{A}$, with P of the above form has a bounded spectrum that is independent of the mesh width in the finite element discretization.

However, indefinite (constraint) preconditioners, see e.g. [5, 10, 7], were not considered by the authors of [3]. We consider the following two indefinite preconditioners

$$P_{con_D} = \begin{bmatrix} A_{\Omega_2} & 0 & 0 \\ 0 & A_{\Omega_1} & B^T \\ 0 & B & 0 \end{bmatrix}, P_{con_T} = \begin{bmatrix} A_{\Omega_2} & 0 & 0 \\ -A_{\Gamma_{12}} & A_{\Omega_1} & B^T \\ 0 & B & 0 \end{bmatrix}$$

The theory presented in [6] does not include indefinite preconditioners and therefore we extend this theory by proving that the spectrum of the preconditioned operator $P^{-1}\mathcal{A}$, where P is of the form (3), is bounded independently of the finite element discretization parameter.

Recall the following saddle point system

$$\mathcal{A} = \begin{bmatrix} A & C^T \\ C & 0 \end{bmatrix} \tag{3.1}$$

where $A \in \mathbb{R}^{n_1 \times n_1}$, $C \in \mathbb{R}^{n_2 \times n_1}$. It is assumed that the matrix $\mathcal{A}$ satisfies the following stability conditions

$$\max_{w \in \mathbb{R}^n \setminus \{0\}} \max_{v \in \mathbb{R}^n \setminus \{0\}} \frac{w^T \mathcal{A} v}{\|w\|_H \|v\|_H} \leq c_1, \tag{3.2a}$$

$$\min_{w \in \mathbb{R}^n \setminus \{0\}} \max_{v \in \mathbb{R}^n \setminus \{0\}} \frac{w^T \mathcal{A} v}{\|w\|_H \|v\|_H} \geq c_2. \tag{3.2b}$$

With the aid of the following lemmas (proven in [6]) we show that the constraint preconditioner

$$P_{con} = \begin{bmatrix} P_1 & C^T \\ C & 0 \end{bmatrix} \tag{3.3}$$

is H-norm equivalent to the operator $\mathcal{A}$ where

$$H = \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix} \tag{3.4}$$

and H_1, H_2 are symmetric positive definite matrices.

Definition 3.1. *Two nonsingular matrices $M, N \in \mathbb{R}^{n \times n}$ are H-norm equivalent if there exists α, β independent of n such that*

$$\alpha \leq \frac{\|Mx\|_H}{\|Nx\|_H} \leq \beta \tag{3.5}$$

and we write $M \sim_H N$.

As a consequence, if $M \sim_H N$ then

$$\|MN^{-1}\|_H \leq \beta \tag{3.6a}$$

$$\|NM^{-1}\|_H \leq \alpha^{-1} \tag{3.6b}$$

It can be shown that H-norm equivalence is an equivalence relation.

Definition 3.2. *Let $M \in \mathbb{R}^{m \times n}$ and $H_1 \in \mathbb{R}^{n \times n}$, $H_2 \in \mathbb{R}^{m \times m}$ two symmetric positive definite matrices, then*

$$\|M\|_{H_1, H_2} = \max_{v \in \mathbb{R}^n \setminus \{0\}} \frac{\|Mv\|_{H_2}}{\|v\|_{H_1}}.$$

Moreover, we have the following useful set of equalities

$$\|H_2^{-1/2} M H_1^{-1/2}\|_2 = \|M\|_{H_1, H_2^{-1}} = \|MH_1^{-1}\|_{H_1^{-1}, H_2^{-1}} = \|H_2^{-1} M\|_{H_1, H_2}. \tag{3.7}$$

Here $\|\cdot\|_2$ denotes the matrix norm induced by the 2 inner product. We state the following lemmas proven by Loghin and Wathen [6].

Lemma 3.1. *Let* (3.2) *hold, then* $H \sim_{H^{-1}} \mathcal{A}$ *and* $H^{-1} \sim_H \mathcal{A}^{-1}$ *and in particular*

$$\|H^{-1}\mathcal{A}\|_H = \|\mathcal{A}H^{-1}\|_{H^{-1}} \leq c_1, \tag{3.8a}$$

$$\|\mathcal{A}^{-1}H\|_H = \|H\mathcal{A}^{-1}\|_{H^{-1}} \leq c_2^{-1}. \tag{3.8b}$$

Lemma 3.2. *Let* (3.2) *hold and assume* $P \sim_{H^{-1}} H$*, then*

$$P \sim_{H^{-1}} \mathcal{A} \quad \textit{and} \quad P^{-1} \sim_H \mathcal{A}^{-1}.$$

Lemma 3.3. *Let* (3.2) *hold then* $\|A\|_{H_1,H_1^{-1}} \leq c_1$, $\|C\|_{H_1,H_2^{-1}} \leq c_1$.

Lemma 3.4. *Let* (3.2) *hold. If there exists* c_3 *independent of* n *such that*

$$\min_{w\in\mathbb{R}^n\setminus\{0\}} \max_{v\in\mathbb{R}^n\setminus\{0\}} \frac{w^T A v}{\|w\|_{H_1}\|v\|_{H_1}} \geq c_3,$$

then $S = BA^{-1}B^T$*, the negative Schur complement, satisfies* $S \sim_{H_2^{-1}} H_2$ *and* $H_2^{-1} \sim_{H_2} S^{-1}$*. Hence there exists* c_4 *such that* $\|S^{-1}\|_{H_2^{-1},H_2} \leq c_4$

Lemma 3.5.

$$\|M\|_{H_1,H_2^{-1}} = \|M^T\|_{H_2,H_1^{-1}}$$

We now state the theorem on the spectral equivalence of constraint preconditioners.

Theorem 3.1. *Let* P_{con} *be defined as in* (3.3)*. Furthermore let* (3.2) *and Lemma* 3.4 *hold. If* $P_1 \sim_{H_1^{-1}} H_1$*, then* $P_{con} \sim_{H^{-1}} \mathcal{A}$ *and* $P_{con}^{-1} \sim_H \mathcal{A}^{-1}$*.*

Proof. We prove that $P_{con} \sim_{H^{-1}} \mathcal{A}$ since $P_{con}^{-1} \sim_H \mathcal{A}^{-1}$ follows similarly. By Lemma 3.2 we need only show that $P_{con} \sim_{H^{-1}} H$ because then by transitivity the result is proven. To prove the above equivalence we bound both $\|H^{-1/2}P_{con}H^{-1/2}\|_2$ and $\|H^{1/2}P_{con}^{-1}H^{1/2}\|_2$. By the assumption that $P_1 \sim_{H_1^{-1}} H_1$ we know there exists β_1 such that $\|P_1H_1^{-1}\|_{H_1^{-1}} \leq \beta_1$. Now consider

$$H^{-1/2}P_{con}H^{-1/2} = \begin{bmatrix} H_1^{-1/2}P_1H_1^{-1/2} & H_1^{-1/2}C^TH_2^{-1/2} \\ H_2^{-1/2}CH_1^{-1/2} & 0 \end{bmatrix}, \tag{3.9}$$

We can bound the two norm of the above matrix as follows

$$\begin{aligned} \|H^{-1/2}P_{con}H^{-1/2}\|_2 &\leq \|H_1^{-1/2}P_1H_1^{-1/2}\|_2 + \|H_2^{-1/2}CH_1^{-1/2}\|_2 + \|H_1^{-1/2}C^TH_2^{-1/2}\|_2, \\ &= \|H_1^{-1}P_1\|_{H_1} + \|C\|_{H_1,H_2^{-1}} + \|C^T\|_{H_2,H_1^{-1}}, \\ &\leq \beta_1 + c_1 + c_1. \end{aligned}$$

The first inequality is a result of expressing the block matrix as a sum of outer products and using that the norm of the entire sum is bounded above by the norm of each summand. The equality in the second line is obtained by using (3.7). Lastly, the final bound comes from (3.2) and the fact that $\|H_1^{-1}P_1\|_{H_1} = \|P_1H_1^{-1}\|_{H_1^{-1}} \leq \beta_1$.

The inverse of P_{con} is

$$P_{con}^{-1} = \begin{bmatrix} P_1^{-1} + P_1^{-1}C^TS^{-1}CP_1^{-1} & -P_1^{-1}C^TS^{-1} \\ -S^{-1}CP_1^{-1} & S^{-1} \end{bmatrix},$$

and therefore

$$H^{1/2}P_{con}^{-1}H^{1/2} = \begin{bmatrix} H_1^{1/2}(P_1^{-1} + P_1^{-1}C^TS^{-1}CP_1^{-1})H_1^{1/2} & -H_1^{1/2}(P_1^{-1}C^TS^{-1})H_2^{1/2} \\ -H_2^{1/2}(S^{-1}CP_1^{-1})H_1^{1/2} & H_2^{1/2}S^{-1}H_2^{1/2} \end{bmatrix}.$$

Hence $\|H^{1/2}P_{con}^{-1}H^{1/2}\|_2$ can be bounded by bounding the following five terms

$$\begin{aligned}
(I) &= \|H_1^{1/2}P_1^{-1}H_1^{1/2}\|_2 \\
(II) &= \|H_1^{1/2}P_1^{-1}C^TS^{-1}CP_1^{-1}H_1^{1/2}\|_2 \\
(III) &= \|H_1^{1/2}P_1^{-1}C^TS^{-1}H_2^{1/2}\|_2 \\
(IV) &= \|H_2^{1/2}S^{-1}CP_1^{-1}H_1^{1/2}\|_2 \\
(V) &= \|H_2^{1/2}S^{-1}H_2^{1/2}\|_2
\end{aligned}$$

Note that

$$(I) = \|P_1^{-1}H_1\|_{H_1} \leq \alpha_1^{-1}$$

and

$$(V) = \|S^{-1}\|_{H_2^{-1},H_2} \leq c_4.$$

Further note that

$$\begin{aligned}
(II) &= \|H_1^{1/2}P_1^{-1}H_1^{1/2}H_1^{-1/2}C^TH_2^{-1/2}H_2^{1/2}S^{-1}H_2^{1/2}H_2^{-1/2}CH_1^{-1/2}H_1^{1/2}P_1^{-1}H_1^{1/2}\|_2 \\
&\leq \|H_1^{1/2}P_1^{-1}H_1^{1/2}\|_2\|H_1^{-1/2}C^TH_2^{-1/2}\|_2\|H_2^{1/2}S^{-1}H_2^{1/2}\|_2\|H_2^{-1/2}CH_1^{-1/2}\|_2\|H_1^{1/2}P_1^{-1}H_1^{1/2}\|_2 \\
&\leq \alpha_1^{-1}c_1c_4c_1\alpha_1^{-1}
\end{aligned}$$

The terms (III) and (IV) are then bounded in a similar manner. □

4 Numerical Results

To illustrate the mesh independent convergence of the constraint preconditioner we consider a numerical experiment where boundary conditions are chosen so that the exact solution is

$$\begin{cases} \mathbf{u}_1(x,y) = \left[-\cos\left(\frac{\pi}{2}y\right)\sin\left(\frac{\pi}{2}x\right) + 1.0, \sin\left(\frac{\pi}{2}y\right)\cos\left(\frac{\pi}{2}x\right) - 1.0 + x\right]^T, \\ p_1(x,y) = 1 - x, \\ p_2(x,y) = \frac{2}{\pi}\cos\left(\frac{\pi}{2}x\right)\cos\left(\frac{\pi}{2}y\right) - y(x-1). \end{cases} \tag{4.1}$$

Note that $\mathbf{K} = \mathbf{I}$ and $\nu = 1$. We refer to problems where the Darcy flow is approximated by continuous basis functions as CGCG and when the Darcy flow is approximated using discontinuous galerkin methods as CGDG. We also take $\rho = 0.6$ as is done in [3] and the stopping criterion for GMRES is when $\|r_k\|/\|r_0\| < 10^{-10}$.

Figure 2 contains plots of the residual convergence curves for increasing degrees of freedom (dof). Note that as the degrees of freedom increase, corresponding to a decrease in the underlying mesh size, the number of iterations does not increase for the constraint preconditioner. Table 1 contains iteration counts and CPU times for both the CGCG and CGDG cases. In both cases the constraint preconditioner outperforms the other preconditioners both with respect to iteration counts and CPU times.

Table 1: Number of iterations for convergence and CPU times

(a) CGCG

DOF	P_+	$P_{T_1}(0.6)$	$P_{T_2}(0.6)$	$P_C(0.6)$	P_{con_D}	P_{con_T}
8225	73 (0.8601)	74 (0.9022)	49 (0.5586)	32 (0.3477)	7 (0.1271)	4 (0.1433)
32833	64 (3.7133)	66 (4.0472)	42 (2.5609)	26 (1.5992)	7 (0.7375)	4 (1.0033)
131201	56 (19.067)	58 (21.262)	36 (12.853)	20 (7.5179)	7 (4.3489)	4 (6.3644)
524545	41 (70.663)	44 (79.163)	27 (48.709)	14 (26.406)	7 (22.886)	4 (40.956)

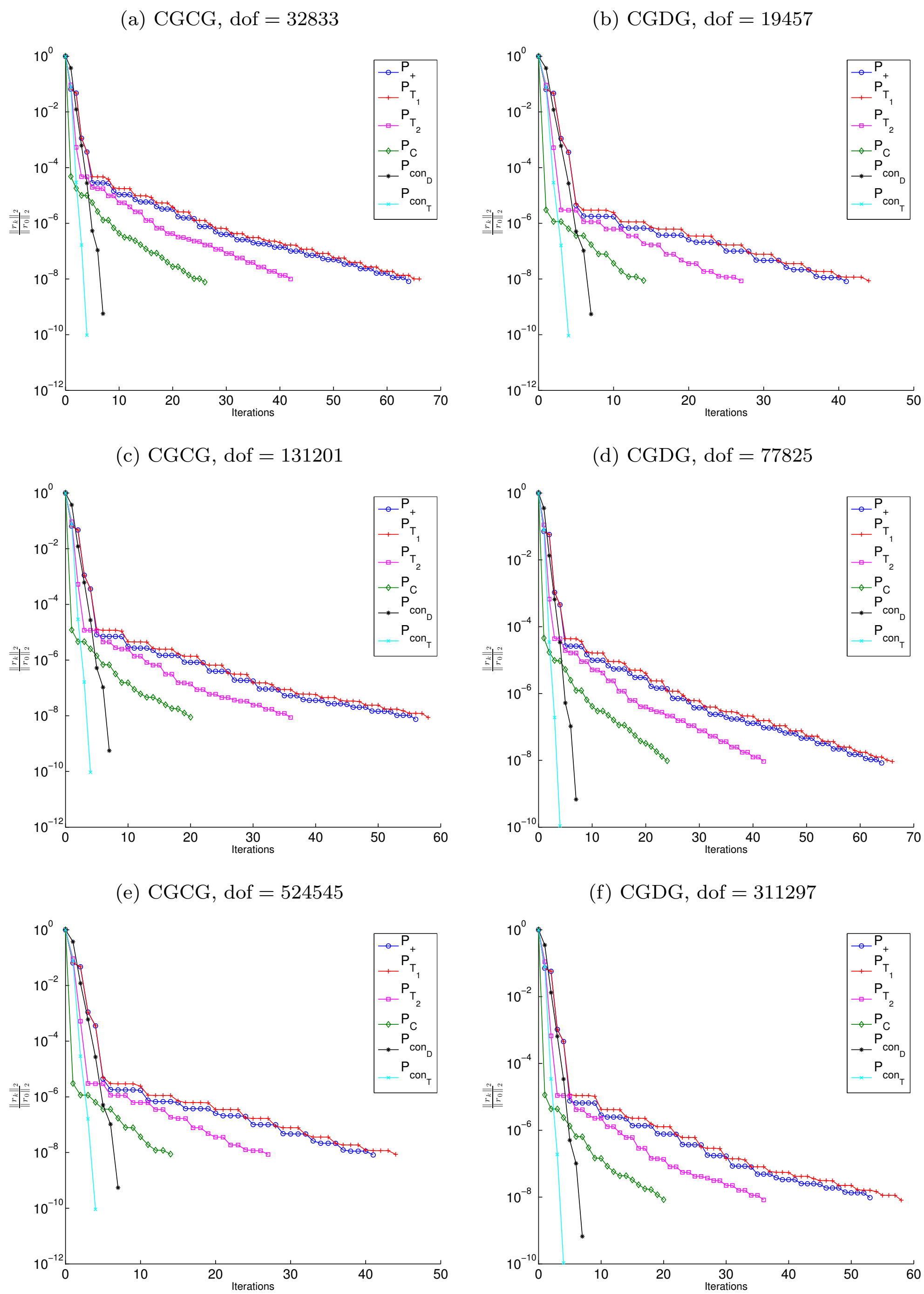
(b) CGDG

DOF	P_+	$P_{T_1}(0.6)$	$P_{T_2}(0.6)$	$P_C(0.6)$	P_{con_D}	P_{con_T}
4865	70 (0.7609)	61 (0.6183)	46 (0.4539)	34 (0.3314)	7 (0.0778)	4 (0.0494)
19457	72 (4.0904)	75 (4.2966)	50 (2.8791)	31 (1.8031)	7 (0.4663)	4 (0.2960)
77825	64 (19.579)	66 (20.410)	42 (12.255)	24 (7.1763)	7 (2.6145)	4 (1.5788)
311297	53 (87.258)	58 (96.030)	36 (57.584)	20 (33.157)	7 (13.999)	4 (7.9819)

5 Conclusions

We have examined the performance of several different preconditioners, standard block diagonal and block triangular as well as constraint preconditioners for the solution of the coupled Stokes-Darcy system. We have proved bounds on the spectrum of the preconditioned operator for the constraint preconditioner showing that it is independent of the mesh discretization parameter. The experiments presented further illustrate this mesh independent convergence. Moreover our experiments also show that in terms of CPU time, the constraint preconditioner is the better choice.

Figure 2: Residual convergence curves of preconditioned GMRES for increasing degrees of freedom for Problem 4.1



References

[1] G. S. Beavers and D. D. Joseph. Boundary conditions at a naturally permeable wall. *Journal of Fluid Mechanics*, 30:197–207, 1967.

[2] M. Benzi, G. H. Golub, and J. Liesen. Numerical solution of saddle point problems. *Acta numerica*, 14:1–137, 2005.

[3] M. Cai, M. Mu, and J. Xu. Preconditioning techniques for a mixed Stokes/Darcy model in porous media applications. *Journal of computational and applied mathematics*, 233:346–355, 2009.

[4] P. Chidyagwai and B. Rivière. Numerical modelling of coupled surface and subsurface flow systems. *Advances in Water Resources*, 33:92–105, 2010.

[5] C. Keller, N. I. Gould, and A. J. Wathen. Constraint preconditioning for indefinite linear systems. *SIAM Journal on Matrix Analysis and Applications*, 21:1300–1317, 2000.

[6] D. Loghin and A. J. Wathen. Analysis of preconditioners for saddle-point problems. *SIAM Journal on Scientific Computing*, 25:2029–2049, 2004.

[7] I. Perugia and V. Simoncini. Block-diagonal and indefinite symmetric preconditioners for mixed finite element formulations. *Numerical linear algebra with applications*, 7:585–616, 2000.

[8] Y. Saad and M. H. Schultz. GMRES: A generalized minimal residual algorithm for solving nonsymmetric linear systems. *SIAM Journal on Scientific and Statistical Computing*, 7:856–869, 1986.

[9] P. G. Saffman. On the boundary condition at the surface of a porous medium. *Studies in Applied Mathematics*, 50:93–101, 1971.

[10] D. Sesana and V. Simoncini. Spectral analysis of inexact constraint preconditioning for symmetric saddle point matrices. *Linear Algebra and its Applications*, 438:2683–2700, 2013.