A SHARP BOUND ON THE CONVERGENCE RATE OF AN AGGREGATION-BASED ALGEBRAIC MULTI-GRID METHOD APPLIED TO A 1D MODEL PROBLEM

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Abstract. We consider the linear system $Ax = b$ arising from one-dimensional Poisson's equation with Dirichlet boundary conditions, where A is the square matrix having the stencil form $\begin{bmatrix}-1 & 2 & -1\end{bmatrix}$. Here we show, using some properties of centrosymmetric matrices, that a pairwise aggregation-based algebraic 2-grid method reduces the A-norm of the error at each step by at least the factor $1/\sqrt{2}$.

1. Introduction

Notations. $||\cdot||$ denotes the 2-norm; For any positive definite symmetric matrix A, the A-norms of a vector x and a matrix G are defined as $||x||_A = ||A^{1/2}x||$ and $||G||_A = ||A^{1/2}GA^{-1/2}||$, respectively; $\langle x, y\rangle$ is the inner product $\sum x_j y_j$.

A 1D Poisson's problem with Dirichlet boundary conditions induces the linear system $Ax = b$, where A is the the N by N matrix $\begin{bmatrix}-1 & 2 & -1\end{bmatrix}$. The authors in [3] show that a pairwise aggregation-based algebraic 2-grid method reduces the A-norm of the error at each step by at least the factor $\sqrt{5/8}$. Numerical computations, however, show that the actual reduction factor is $1/\sqrt{2}$. In this paper, we will show that the expected factor $1/\sqrt{2}$ is theoretically correct.

2. Analysis of the A-norm of the error

Assuming that N is even, define the N by $N/2$ piecewise constant prolongation matrix P by $P_{2l-1,l} = P_{2l,l} = 1$ for $l = 1, 2, \ldots, N/2$ with all other entries of P being 0 and an $N/2$ by $N/2$ coarse grid matrix A_C by $A_C = P^TAP$ (Galerkin condition). After a coarse grid solve followed by a weighted Jacobi iteration, the authors in [3] show the relation

$$||e_{j+1}||_A \leq \sigma ||e_j||_A \tag{1}$$

for the error $e_j = x - x_j$, where

$$\sigma = ||(I - \frac{1}{4}A)(I - A^{1/2}PA_C^{-1}P^TA^{1/2})||$$

(I is the identity matrix). The procedure is as follows:

(a) Coarse grid solve: $A_C\delta_C = P^Tr_j$, where $r_j = b - Ax_j$ is the residual on the fine grid.

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(b) Correction: $x'_j = x_j + P\delta_C = x_j + Cr_j$, where $C = PA_C^{-1}P^T$, and its corresponding residual is $r'_j = b - Ax'_j = (I - AC)r_j$.

(c) Relaxation: With the weighted Jacobi iteration with damping factor of 2,

$$\begin{aligned} x_{j+1} &= x'_j + (2\text{diag}(A))^{-1}r'_j = x'_j + \frac{1}{4}r'_j \\ r_{j+1} &= b - Ax_{j+1} = (I - \frac{1}{4}A)(I - AC)r_j \\ e_{j+1} &= A^{-1}r_{j+1} = (I - \frac{1}{4}A)(I - CA)e_j \end{aligned}$$

(d) A-norm of the error: Since

$$\begin{aligned} ||(I - \frac{1}{4}A)(I - CA)||_A &= ||A^{1/2}(I - \frac{1}{4}A)(I - CA)A^{-1/2}|| \\ &= ||(I - \frac{1}{4}A)(I - A^{1/2}CA^{1/2})||, \end{aligned}$$

we have the inequality (1).

Moreover, they also show an upper bound for σ as follows:

(a) Following the approach in [2, Ch 12],

$$\sigma \leq \max_{\substack{||y||=||I-A^{1/2}CA^{1/2}|| \\ y\in\mathcal{R}(I-A^{1/2}CA^{1/2})}} ||(I - \frac{1}{4}A)y||,$$

where $\mathcal{R}(\cdot)$ denotes the range. Theorem 12.1.1 in [2] shows that

$$\begin{aligned} ||I - A^{1/2}CA^{1/2}|| &= 1 \\ \mathcal{R}(I - A^{1/2}CA^{1/2}) &= A^{-1/2}\mathcal{N}(P^T), \end{aligned}$$

where $\mathcal{N}(\cdot)$ denotes the null space. Thus,

$$\sigma \leq \max_{\substack{||y||=1 \\ y\in A^{-1/2}\mathcal{N}(P^T)}} ||(I - \frac{1}{4}A)y||$$

(b) The eigenvalues and orthonormal eigenvectors of A are:

$$\begin{aligned} \lambda_k &= 2 - 2\cos\frac{k\pi}{N+1}, \quad k = 1, \ldots, N \\ q_j^{(k)} &= \sqrt{\frac{2}{N+1}}\sin\frac{jk\pi}{N+1}, \quad j, k = 1, \ldots, N \end{aligned}$$

Let $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_N)$ and $Q = \begin{pmatrix} q^{(1)} & \cdots & q^{(N)} \end{pmatrix}$. Since Q is a symmetric orthogonal matrix and $A = Q\Lambda Q$,

$$\sigma \leq \max_{\substack{||z||=1 \\ z\in\Lambda^{-1/2}Q\mathcal{N}(P^T)}} ||(I - \frac{1}{4}\Lambda)z||$$

(c) $\mathcal{N}(P^T)$ is spanned by the basis $\{\mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_3 - \mathbf{e}_4, \ldots, \mathbf{e}_{N-1} - \mathbf{e}_N\}$, where $\mathbf{e}_j$ is the elementary unit vector with 1 in the jth entry. Therefore,

$$\sigma \leq \max_{\substack{||z||=1 \\ z\in\mathcal{S}}} ||(I - \frac{1}{4}\Lambda)z||,$$

where $\mathcal{S}$ is the space spanned by the vectors

$$\Lambda^{-1/2}(q^{(1)} - q^{(2)}), \Lambda^{-1/2}(q^{(2)} - q^{(3)}), \dots, \Lambda^{-1/2}(q^{(N-1)} - q^{(N)}).$$

Therefore, to prove $||e_{j+1}||_A \le \frac{1}{\sqrt{2}}||e_j||_A$, it is enough to show

$$\max_{\substack{||z||=1 \\ z\in\mathcal{S}}} ||(I - \frac{1}{4}\Lambda)z||^2 \le \frac{1}{2} \tag{2}$$

We will use the following three lemmas to prove (2) in Theorem 4.

Lemma 1. *Let* $x_k = k\pi/(N+1)$.

(a) For any integer n,

$$\sum_{k=1}^{N} \cos(nx_k) = \begin{cases} 0, & \text{if } n \text{ is odd,} \\ N, & \text{if } n \text{ is a multiple of } 2(N+1), \\ -1, & \text{otherwise} \end{cases} \tag{3}$$

(b) For any integers l and m,

$$\begin{aligned} & (\sin(2l-1)x_k - \sin(2lx_k))(\sin(2m-1)x_k - \sin(2mx_k)) \\ = \; & (1-\cos x_k)\left[\cos((2l+2m-1)x_k) + \cos(2(l-m)x_k)\right] \end{aligned} \tag{4}$$

(c) For any integer n,

$$\sum_{k=1}^{N} (1-\cos x_k)^2 \cos(nx_k) = \begin{cases} (3N-1)/2, & \text{if } n=0 \\ -N+1, & \text{if } n=1 \\ (N-7)/4, & \text{if } n=2 \\ 2(-1)^{n+1}, & \text{otherwise} \end{cases} \tag{5}$$

Proof. Since $\cos(nx_{N+1-k}) = \cos(n\pi - x_k) = (-1)^n \cos(x_k)$ for any k,

$$\sum_{k=1}^{N} \cos(nx_k) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 2\sum_{k=1}^{N/2} \cos(nx_k) & \text{if } n \text{ is even} \end{cases}.$$

If n is a multiple of $2(N+1)$, then $\cos(nx_k) = 1$ for any k and thus $\sum_{k=1}^{N} \cos(nx_k) = N$. If n is an even integer, not being a multiple of $2(N+1)$, then using the known formula

$$\sum_{k=1}^{N} \cos(k\theta) = \frac{-1}{2} + \frac{\sin(N+\frac{1}{2})\theta}{2\sin\frac{1}{2}\theta},$$

we have

$$\sum_{k=1}^{N/2} \cos(nx_k) = \frac{-1}{2} + \frac{\sin(N+1)\frac{n\pi}{2(N+1)}}{2\sin\frac{n\pi}{2(N+1)}} = \frac{-1}{2}$$

and thus $\sum_{k=1}^{N} \cos(nx_k) = -1$.

The remaining two results can be easily shown using the following basic trigonometric identities:

$$\begin{aligned}
2\sin A\sin B &= \cos(A-B)-\cos(A+B)\\
\cos A+\cos B &= 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}\\
2\cos^2 x &= 1+\cos(2x)\\
4\cos^3 x &= \cos(3x)+3\cos x
\end{aligned}$$

□

Lemma 2. *For $l, m = 1, \dots, N/2$, we have*

$$\langle \Lambda^{-1/2}(q^{(2l-1)} - q^{(2l)}),\, \Lambda^{-1/2}(q^{(2m-1)} - q^{(2m)})\rangle = \begin{cases} \frac{N}{N+1}, & \text{if } l = m,\\ \frac{-1}{N+1} & \text{if } l \neq m \end{cases} \tag{6}$$

(See also [3]*) and*

$$\langle \Lambda^{1/2}(q^{(2l-1)} - q^{(2l)}),\, \Lambda^{1/2}(q^{(2m-1)} - q^{(2m)})\rangle = \begin{cases} 6, & \text{if } l = m,\\ 1, & \text{if } l - m = \pm 1,\\ 0, & \text{otherwise} \end{cases} \tag{7}$$

Proof. Let $x_k = \frac{k\pi}{N+1}$. Then,

$$\begin{aligned}
&\langle \Lambda^{-1/2}(q^{(2l-1)} - q^{(2l)}),\, \Lambda^{-1/2}(q^{(2m-1)} - q^{(2m)})\rangle\\
&= \sum_{k=1}^{N} \lambda_k^{-1/2}(q_k^{(2l-1)} - q_k^{(2l)}) \cdot \lambda_k^{-1/2}(q_k^{(2m-1)} - q_k^{(2m)})\\
&= \frac{1}{N+1}\sum_{k=1}^{N}\cos((2l+2m-1)x_k) + \frac{1}{N+1}\sum_{k=1}^{N}\cos(2(l-m)x_k), \text{ by (4).}
\end{aligned}$$

By (3), the first sum in the right hand side is 0 for any l, m; meanwhile,

$$\sum_{k=1}^{N}\cos(2(l-m)x_k) = \begin{cases} N, & \text{if } l = m,\\ -1, & \text{otherwise} \end{cases}.$$

Thus the result (6) follows. Similarly, using (4),

$$\begin{aligned}
&\langle \Lambda^{1/2}(q^{(2l-1)} - q^{(2l)}),\, \Lambda^{1/2}(q^{(2m-1)} - q^{(2m)})\rangle\\
&= \sum_{k=1}^{N} \lambda_k^{1/2}(q_k^{(2l-1)} - q_k^{(2l)}) \cdot \lambda_k^{1/2}(q_k^{(2m-1)} - q_k^{(2m)})\\
&= \frac{4}{N+1}\sum_{k=1}^{N}(1-\cos x_k)^2\left[\cos((2l+2m-1)x_k) + \cos(2(l-m)x_k)\right]
\end{aligned}$$

and the result (7) follows from (5). □

Let $L = N/2$. To prove (2), it is enough to show that $||(I - \frac{1}{4}\Lambda)z||^2 \leq \frac{1}{2}$ for any z such that z is a unit vector of the form $\sum_{l=1}^{L} c_l \Lambda^{-1/2}(q^{(2l-1)} - q^{(2l)})$.

Lemma 3. *For $z = \sum_{l=1}^{L} c_l \Lambda^{-1/2}(q^{(2l-1)} - q^{(2l)})$, the constraint $||z|| = 1$ is equivalent to*

$$\sum_{l=1}^{L} c_l^2 = 1 + \frac{1}{2L} + \frac{1}{L} \sum_{1 \le l < m \le L} c_l c_m \tag{8}$$

and the inequality $||(I - \frac{1}{4}\Lambda)z||^2 \le \frac{1}{2}$ is equivalent to

$$\sum_{l=1}^{L} c_l^2 - \frac{1}{5} \sum_{l=2}^{L} c_{l-1} c_l \ge \frac{4}{5}. \tag{9}$$

Proof. Since

$$\begin{aligned} ||z||^2 &= \sum_{l,m} c_l c_m \langle \Lambda^{-1/2}(q^{(2l-1)} - q^{(2l)}), \Lambda^{-1/2}(q^{(2m-1)} - q^{(2m)}) \rangle \\ &= \frac{2L}{2L+1} \sum_{l=1}^{L} c_l^2 - \frac{2}{2L+1} \sum_{1 \le l < m \le L} c_l c_m \end{aligned}$$

by (6), the constraint $||z|| = 1$ is equivalent to (8). Meanwhile, since

$$\langle z, \Lambda z \rangle = \sum_{l,m} c_l c_m \langle (q^{(2l-1)} - q^{(2l)}), (q^{(2m-1)} - q^{(2m)}) \rangle = 2 \sum_{l=1}^{L} c_l^2$$

by the orthonormal property of the vectors $q^{(k)}$ and

$$\begin{aligned} ||\Lambda z||^2 &= \sum_{l,m} c_l c_m \langle \Lambda^{1/2}(q^{(2l-1)} - q^{(2l)}), \Lambda^{1/2}(q^{(2m-1)} - q^{(2m)}) \rangle \\ &= 6 \sum_{l=1}^{L} c_l^2 + \sum_{l-m=\pm 1} c_l c_m \end{aligned}$$

by (7), we have

$$\begin{aligned} ||(I - \frac{1}{4}\Lambda)z||^2 &= 1 - \frac{1}{2}\langle z, \Lambda z \rangle + \frac{1}{16}||\Lambda z||^2 \\ &= 1 - \frac{5}{8} \sum_{l=1}^{L} c_l^2 + \frac{1}{8} \sum_{l=2}^{L} c_{l-1} c_l \end{aligned}$$

and thus the inequality $||(I - \frac{1}{4}\Lambda)z||^2 \le \frac{1}{2}$ is equivalent to (9). □

Theorem 4. *The inequality (9) subject to the constraint (8) holds for any positive integer L.*

Proof. Define two functions $f(c)$ and $g(c)$ by

$$\begin{aligned} f(c) &= \sum_{l=1}^{L} c_l^2 - \frac{1}{5} \sum_{l=2}^{L} c_{l-1} c_l, \\ g(c) &= \sum_{l=1}^{L} c_l^2 - 1 - \frac{1}{2L} - \frac{1}{L} \sum_{1 \le l < m \le L} c_l c_m. \end{aligned}$$

We will use the method of Lagrange multipliers to show that $f(c) \geq \frac{4}{5}$ subject to the condition $g(c) = 0$. The constrained minimum occurs at points c such that $\nabla_c f + \lambda \nabla_c g = 0$ for some λ. The equation can be expressed by

$$(2I - \frac{1}{5}R + \lambda(2I - \frac{1}{L}S))c = 0$$

or equivalently

$$(\frac{1}{5}R + \frac{\lambda}{L}S)c = 2(1+\lambda)c,$$

where

$$R = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & \ddots & \\ & \ddots & 0 & 1 \\ & & 1 & 0 \end{pmatrix} \text{ and } S = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \ddots & 1 \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & 0 \end{pmatrix}.$$

A matrix X is called centrosymmetric if $XJ = JX$, where J is the square matrix with 1 on the counterdiagonal and 0 elsewhere. Since two matrices R and J are symmetric and centrosymmetric, the equation $(\frac{1}{5}R+\frac{\lambda}{L}S)c = 2(1+\lambda)c$ implies that c is an eigenvector of the symmetric centrosymmetric matrix $\frac{1}{5}R+\frac{\lambda}{L}S$. According to [1], such a vector c is either symmetric (that is, $c_{L+1-k} = c_k$ for all $k = 1, \ldots, L/2$) or skew symmetric (that is, $c_{L+1-k} = -c_k$ for all $k = 1, \ldots, L/2$).

Case 1: Assume that L is a power of 2. In the ensuing computations, we will use the following results (explicitly or implicitly): for any even integer M,

$$\sum_{1\leq l<m\leq M} c_l c_m = \begin{cases} -\sum_{l=1}^{M/2} c_l^2, & \text{when } c \text{ is skew symmetric} \\ \sum_{l=1}^{M/2} c_l^2 + 4\sum_{1\leq l<m\leq M/2} c_l c_m, & \text{when } c \text{ is symmetric} \end{cases}$$

When c is skew symmetric, the constraint $g(c) = 0$ is equivalent to $\sum_{l=1}^{L/2} c_l^2 = \frac{1}{2}$ and the inequality $f(c) \geq \frac{4}{5}$ is equivalent to $2\sum_{l=2}^{L/2} c_{l-1}c_l - c_{L/2}^2 \leq 1$. Since

$$2\sum_{l=2}^{L/2} c_{l-1}c_l - c_{L/2}^2 \leq c_1^2 + 2\sum_{l=2}^{L/2-1} c_l^2 \leq 2\sum_{l=1}^{L/2} c_l^2,$$

the inequality $f(c) \geq \frac{4}{5}$ is true for any c such that $g(c) = 0$. Meanwhile, when c is symmetric, the constraint $g(c) = 0$ is equivalent to

$$(L - \frac{1}{2})\sum_{l=1}^{L/2} c_l^2 = \frac{2L+1}{2^2} + 2\sum_{1\leq l<m\leq L/2} c_l c_m. \tag{10}$$

Using the identity $\sum_{l=2}^{L} c_{l-1}c_l = c_{L/2}^2 + 2\sum_{l=2}^{L/2} c_{l-1}c_l$, we can show that the inequality $f(c) \geq \frac{4}{5}$ is expressed by

$$2\sum_{l=1}^{L/2} c_l^2 - \frac{1}{5}\left(c_{L/2}^2 + 2\sum_{l=2}^{L/2} c_{l-1}c_l\right) \geq \frac{4}{5}.$$

Moreover, since

$$2\sum_{l=1}^{L/2} c_l^2 - \frac{1}{5}\left(c_{L/2}^2 + 2\sum_{l=2}^{L/2} c_{l-1}c_l\right) \geq 2\sum_{l=1}^{L/2} c_l^2 - \frac{2}{5}\sum_{l=1}^{L/2} c_l^2 = \frac{8}{5}\sum_{l=1}^{L/2} c_l^2,$$

it is enough to show that $\sum_{l=1}^{L/2} c_l^2 \geq \frac{1}{2}$ subject to the constraint (10). Now we have another minimization problem with constraints. Using the method of Lagrange multipliers and the results in [1] again, the vector c of size $L/2$ is either symmetric or skew symmetric. When c is skew symmetric, the constraint (10) is equivalent to $\sum_{l=1}^{L/2^2} c_l^2 = \frac{1}{2^2}$ and the inequality $\sum_{l=1}^{L/2} c_l^2 \geq \frac{1}{2}$ is just $\sum_{l=1}^{L/2^2} c_l^2 \geq \frac{1}{2^2}$. Thus, the inequality subject to the constraint is clear. In the case that c is symmetric, we need to show $\sum_{l=1}^{L/2^2} c_l^2 \geq \frac{1}{2^2}$ when c satisfies that

$$(L - \frac{3}{2}) \sum_{l=1}^{L/2^2} c_l^2 = \frac{2L+1}{2^3} + 2^2 \sum_{1 \leq l < m \leq L/2^2} c_l c_m.$$

We can repeat this process. That is, assume that we want to show the inequality

$$\sum_{l=1}^{L/2^n} c_l^2 \geq \frac{1}{2^n} \tag{11}$$

subject to the constraint

$$(L - \frac{2^n - 1}{2}) \sum_{l=1}^{L/2^n} c_l^2 = \frac{2L+1}{2^{n+1}} + 2^n \sum_{1 \leq l < m \leq L/2^n} c_l c_m. \tag{12}$$

For a skew symmetric c, the constraint is $\sum_{l=1}^{L/2^{n+1}} c_l^2 = \frac{1}{2^{n+1}}$ and thus the inequality $\sum_{l=1}^{L/2^n} c_l^2 \geq \frac{1}{2^n}$ is clear. For a symmetric c, the constraint above is equivalent to

$$(L - \frac{2^{n+1} - 1}{2}) \sum_{l=1}^{L/2^{n+1}} c_l^2 = \frac{2L+1}{2^{n+2}} + 2^{n+1} \sum_{1 \leq l < m \leq L/2^{n+1}} c_l c_m$$

and the inequality $\sum_{l=1}^{L/2^n} c_l^2 \geq \frac{1}{2^n}$ is just $\sum_{l=1}^{L/2^{n+1}} c_l^2 \geq \frac{1}{2^{n+1}}$. Consequently, by induction on n, it is enough to find a positive integer n such that the inequality (11) holds for all c satisfying (12). Let $L = 2^p$. Then, when $n = p$, the inequality (11) is equivalent to $2^p c_1^2 \geq 1$ and the constraint (12) is expressed by

$$(2^p - \frac{2^p - 1}{2}) c_1^2 = 1 + \frac{1}{2^{p+1}}.$$

The inequality subject to the constraint $2^p c_1^2 \geq 1$ is clear by the following argument:

$$2^p c_1^2 = \frac{2^p - 1}{2} c_1^2 + 1 + \frac{1}{2^{p+1}} \geq 1.$$

Case 2: Assuming that L is an odd integer, we will show that $f(c) \geq \frac{4}{5}$ subject to the condition $g(c) = 0$, where f, g are defined in the beginning of this proof. Let $L = 2u + 1$. In the case that c is skew symmetric, we have $\sum_{1 \leq l < m \leq L} c_l c_m = -\sum_{l=1}^{u} c_l^2$ and thus the constraint $g(c) = 0$ is equivalent to $\sum_{l=1}^{u} c_l^2 = \frac{1}{2}$. Moreover, since $\sum_{l=2}^{L} c_{l-1} c_l = 2 \sum_{l=2}^{u} c_{l-1} c_l$, the inequality $f(c) \geq \frac{4}{5}$ is expressed by $2 \sum_{l=2}^{u} c_{l-1} c_l \leq 1$, which is clear on the constraint. Now consider the symmetric case. In this case, using the following identities

$$\begin{aligned}\sum_{1\le l<m\le L} c_l c_m &= \sum_{l=1}^{u} c_l^2 + 4\sum_{1\le l<m\le u} c_l c_m + 2c_{u+1}\sum_{l=1}^{u} c_l,\\ \sum_{l=2}^{L} c_{l-1}c_l &= 2\sum_{l=2}^{u+1} c_{l-1}c_l,\end{aligned}$$

we can show that the inequality $f(c) \ge \frac{4}{5}$ is equivalent to

$$2\sum_{l=1}^{u} c_l^2 + c_{u+1}^2 - \frac{2}{5}\sum_{l=2}^{u+1} c_{l-1}c_l \ge \frac{4}{5}.$$

Moreover, by the following argument

$$\begin{aligned}& 2\sum_{l=1}^{u} c_l^2 + c_{u+1}^2 - \frac{2}{5}\sum_{l=2}^{u+1} c_{l-1}c_l\\ \ge\ & 2\sum_{l=1}^{u} c_l^2 + c_{u+1}^2 - \frac{1}{5}(c_{u+1}^2 + 2\sum_{l=1}^{u} c_l^2)\\ =\ & \frac{8}{5}\sum_{l=1}^{u} c_l^2 + \frac{4}{5}c_{u+1}^2,\end{aligned}$$

it is enough to show that

$$\sum_{l=1}^{u} c_l^2 + \frac{1}{2}c_{u+1}^2 \ge \frac{1}{2} \tag{13}$$

subject to the constraint $g(c) = 0$ which is equivalent to

$$(2-\frac{1}{L})\sum_{l=1}^{u} c_l^2 + c_{u+1}^2 = 1 + \frac{1}{2L} + \frac{4}{L}\sum_{1\le l<m\le u} c_l c_m + \frac{2}{L}c_{u+1}\sum_{l=1}^{u} c_l. \tag{14}$$

Let $s = \sum_{l=1}^{u} c_l$ and $r = \sum_{l=1}^{u} c_l^2$. Then, the constraint (14) can be expressed by $c_{u+1}^2 - 2pc_{u+1} + q = 0$, where $p = \frac{1}{L}s$ and $q = \frac{2L+1}{L}(r+\frac{1}{2}) - \frac{2}{L}s^2$. Since $p \pm \sqrt{p^2-q}$ are the roots of the quadratic equation above, we have $c_{u+1}^2 = 2p^2 - q \pm 2p\sqrt{p^2-q}$. Therefore, the inequality (13) is equivalent to

$$r + p^2 - \frac{1}{2}q \pm p\sqrt{p^2-q} \ge \frac{1}{2},$$

where $p^2 - q \ge 0$. Plugging $p = \frac{1}{L}s$ and $q = \frac{2L+1}{L}(r+\frac{1}{2}) - \frac{2}{L}s^2$, the inequality above is expressed by

$$(1+\frac{1}{2L})s^2 + \frac{s^2}{2L} + \frac{1}{4} - \frac{r}{2} \ge 2\sqrt{(1+\frac{1}{2L})s^2}\sqrt{\frac{s^2}{2L} + \frac{1}{4} - \frac{r}{2}}.$$

Since the inequality is true by the relationship between arithmetic mean and geometric mean, the case that L is odd is solved.

Case 3: Finally, we consider the case $L = 2^p q$, where $q > 1$ is odd. Substituting $n = p$ in (11) and (12), it is enough to show that

$$\sum_{l=1}^{q} c_l^2 \ge \frac{1}{2^p} \tag{15}$$

when

$$(2^p q - \frac{2^p - 1}{2}) \sum_{l=1}^{q} c_l^2 = q + \frac{1}{2^{p+1}} + 2^p \sum_{1 \le l < m \le q} c_l c_m. \tag{16}$$

Let $q = 2r + 1$. When c is skew symmetric, (15) is $2\sum_{l=1}^{r} c_l^2 \ge \frac{1}{2^p}$ and (16) is $\sum_{l=1}^{r} c_l^2 = \frac{1}{2^{p+1}}$. Thus the inequality is clear. Meanwhile, if c is symmetric, then (15) is

$$\sum_{l=1}^{r} c_l^2 + \frac{1}{2} c_{r+1}^2 \ge \frac{1}{2^{p+1}} \tag{17}$$

and (16) is

$$\begin{aligned} & (2^{p+1} q - 2^{p+1} + 1) \sum_{l=1}^{r} c_l^2 + \frac{2^{p+1} q - 2^p + 1}{2} c_{r+1}^2 \\ = \; & q + \frac{1}{2^{p+1}} + 2^{p+2} \sum_{1 \le l < m \le r} c_l c_m + 2^{p+1} c_{r+1} \sum_{l=1}^{r} c_l. \end{aligned} \tag{18}$$

Let $\xi = \sum_{l=1}^{r} c_l$ and $\zeta = \sum_{l=1}^{r} c_l^2$. Then, the constraint can be expressed by $c_{r+1}^2 - 2\alpha c_{r+1} + \beta = 0$, where

$$\begin{aligned} \alpha &= \frac{2^{p+1}}{2L - 2^p + 1} \xi, \\ \beta &= \frac{4L + 2}{2L - 2^p + 1} \zeta - \frac{2L + 1}{2^p (2L - 2^p + 1)} - \frac{2^{p+2}}{2L - 2^p + 1} \xi^2. \end{aligned}$$

Plugging $c_{r+1}^2 = 2\alpha^2 - \beta \pm 2\alpha\sqrt{\alpha^2 - \beta}$, we can show that the inequality (17) is expressed by

$$\begin{aligned} & 2^{2p+1} \xi^2 + 2^{p+1} (2L + 1) \xi^2 + 2^p (2L - 2^p + 1)(\frac{1}{2^{p+1}} - \zeta) \\ \ge \; & 2 \sqrt{2^{2p+1} \xi^2 \left[2^{p+1} (2L + 1) \xi^2 + 2^p (2L - 2^p + 1)(\frac{1}{2^{p+1}} - \zeta) + (2L - 2^p + 1)^2 (\frac{1}{2^{p+1}} - \zeta) \right]} \end{aligned}$$

Let $X = 2^{2p+1}\xi^2$, $Y = 2^{p+1}(2L+1)\xi^2 + 2^p(2L - 2^p + 1)(\frac{1}{2^{p+1}} - \zeta)$, and $Z = (2L - 2^p + 1)^2(\frac{1}{2^{p+1}} - \zeta)$. Then the above inequality is expressed by $X + Y \ge 2\sqrt{X(Y + Z)}$. If $\zeta \ge \frac{1}{2^{p+1}}$, then the inequality (17) is clear. Thus, we may assume that $\frac{1}{2^{p+1}} - \zeta > 0$. In this case, all of X, Y, and Z are nonnegative. The following argument shows that the desired inequality $X + Y \ge 2\sqrt{X(Y + Z)}$ holds:

$$\begin{aligned} & X + Y \ge 2\sqrt{X(Y + Z)} \\ \iff & (X - Y)^2 \ge 4XZ \\ \iff & \left[2\xi^2 + (\frac{1}{2^{p+1}} - \zeta) \right]^2 \ge 8\xi^2 \cdot (\frac{1}{2^{p+1}} - \zeta) \\ \iff & (2\xi^2 - (\frac{1}{2^{p+1}} - \zeta))^2 \ge 0. \end{aligned}$$

□

We have shown that $||e_{j+1}||_A \leq \frac{1}{\sqrt{2}}||e_j||_A$ using post-smoothing only. In that case, the error satisfies $e_{j+1} = (I - \frac{1}{4}A)(I - CA)e_j$, where $C = PA_C^{-1}P^T$. Considering post-smoothing only, the equation for the error becomes $e_{j+1} = (I - CA)(I - \frac{1}{4}A)e_j$. Since

$$\begin{aligned} ||(I - CA)(I - \frac{1}{4}A)||_A &= ||A^{1/2}(I - CA)(I - \frac{1}{4}A)A^{-1/2}|| \\ &= ||(I - A^{1/2}CA^{1/2})(I - \frac{1}{4}A)|| \\ &= ||(I - \frac{1}{4}A)^T(I - A^{1/2}CA^{1/2})^T|| \\ &= ||(I - \frac{1}{4}A)(I - A^{1/2}CA^{1/2})|| \\ &= ||(I - \frac{1}{4}A)(I - CA)||_A, \end{aligned}$$

we have the same relation $||e_{j+1}||_A \leq \frac{1}{\sqrt{2}}||e_j||_A$ for post-smoothing only. Meanwhile, in the case of pre- and post-smoothing, the error satisfies $e_{j+1} = (I - \frac{1}{4}A)(I - CA)(I - \frac{1}{4}A)e_j$ and thus $||e_{j+1}||_A \leq \sigma||e_j||_A$, where $\sigma = ||(I - \frac{1}{4}A)(I - A^{1/2}CA^{1/2})(I - \frac{1}{4}A)||$. The authors in [3, Theorem 4] proved $\sigma \leq \sqrt{\frac{17}{32}}$, but numerical computations suggest the bound $\sigma \leq \frac{1}{2}$. The following simple argument shows $\sigma \leq \frac{1}{\sqrt{2}}$ for pre- and post-smoothing (rather than showing the sharp bound $\sigma \leq \frac{1}{2}$): Since the matrix $(I - \frac{1}{4}A)(I - A^{1/2}CA^{1/2})(I - \frac{1}{4}A)$ is symmetric, its 2-norm is the spectral radius of the matrix. Thus,

$$\begin{aligned} \sigma &= \rho((I - \frac{1}{4}A)(I - A^{1/2}CA^{1/2})(I - \frac{1}{4}A)) \\ &= \rho((I - \frac{1}{4}A)^2(I - A^{1/2}CA^{1/2})) \\ &\leq ||(I - \frac{1}{4}A)^2(I - A^{1/2}CA^{1/2})|| \\ &\leq ||I - \frac{1}{4}A|| \cdot ||(I - \frac{1}{4}A)(I - A^{1/2}CA^{1/2})||. \end{aligned}$$

Furthermore, since $||I - \frac{1}{4}A|| = \rho(I - \frac{1}{4}A) = \max_{1 \leq k \leq N} \frac{1}{2}(1 + \cos\frac{k\pi}{N+1}) \leq 1$,

$$\sigma \leq ||(I - \frac{1}{4}A)(I - A^{1/2}CA^{1/2})|| \leq \frac{1}{\sqrt{2}}.$$

References

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