MULTIGRID SMOOTHERS FOR MAGNETOHYDRODYNAMICS

THOMAS R. BENSON, JAMES H. ADLER, ERIC C. CYR, SCOTT P. MACLACHLAN, AND RAYMOND S. TUMINARO

ABSTRACT. Magnetohydrodynamic models are used to model a wide range of plasma physics applications. The system of partial differential equations that characterizes these models is nonlinear and strongly couples fluid interactions with electromagnetic interactions. As a result, the linear systems that arise from discretization and linearization of the nonlinear problem can be difficult to solve. In this paper, we consider a multigrid preconditioner for GMRES as a solver for these systems. We compare three potential smoothers for this system, two of which are motivated by well-known smoothers for the incompressible fluids system and the other is a new smoother that splits the physics into a magnetics-velocity operator and a Navier-Stokes operator. Results for a two-dimensional, steady-state test problem are shown.

1. INTRODUCTION

Magnetohydrodynamics (MHD) is a model of plasma physics that treats the plasma as a charged fluid in the presence of electromagnetic fields. These models have applications in the study of solar flares, spacecraft propulsion, and simulations of fusion energy (e.g. tokamak reactors) [7]. Mathematically, this model is a coupling of the equations of motion of a fluid (the Navier-Stokes equations) and the equations governing electromagnetic fields (Maxwell's equations). In general, the resulting system of partial differential equations (PDEs) is challenging to solve because it is nonlinear and time-dependent, and models strongly coupled physical interactions [2, 11]. In this paper, we focus on solving the linear systems that arise from the linearization step of a nonlinear solver, such as Newton's Method. Specifically, we are concerned with finding effective multigrid relaxation schemes for this system. We present results for three different smoothers in the context of geometric multigrid, though they can be extended to schemes for algebraic multigrid.

1.1. The Physical Model.

The MHD model that we consider is the one-fluid visco-resistive MHD system [7]. We are concerned with a formulation in the incompressible limit (constant fluid density, ρ). The $\mathbf{B}$-field formulation of the strong form of the partial differential equations written as residuals is:

$$\mathbf{R}_{\mathbf{u}} = \rho\frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u}\cdot\nabla\mathbf{u}) - \nabla\cdot(\mathbf{T}+\mathbf{T}_M), \tag{1}$$

$$R_{\bar{p}} = \rho\nabla\cdot\mathbf{u}, \tag{2}$$

$$\mathbf{R}_{\mathbf{B}} = \frac{\partial \mathbf{B}}{\partial t} - \nabla\times(\mathbf{u}\times\mathbf{B}) + \nabla\times(\frac{\eta}{\mu_0}\nabla\times\mathbf{B}), \tag{3}$$

where the viscous and magnetic stress tensors are

$$\mathbf{T} = -\bar{p}\mathbf{I} + \mu\left[\nabla\mathbf{u} + \nabla\mathbf{u}^T\right], \tag{4}$$

and

$$\mathbf{T}_M = \frac{1}{\mu_0}\mathbf{B}\otimes\mathbf{B} - \frac{1}{2\mu_0}\left\|\mathbf{B}\right\|^2\mathbf{I}, \tag{5}$$

respectively. Here, $\otimes$ defines a tensor product. The constants in the above equations are density (ρ), dynamic viscosity (μ), magnetic resistivity (η), and magnetic permeability of free space (μ_0). The dependent variables – velocity ($\mathbf{u}$), hydrodynamic pressure ($\bar{p}$), and the magnetic field ($\mathbf{B}$) – satisfy the momentum equation (1), the continuity equation (2), and the magnetics evolution equation (3) when $\mathbf{R}_{\mathbf{u}} = \mathbf{0}$, $\mathbf{R}_{\mathbf{B}} = \mathbf{0}$, and $R_{\bar{p}} = 0$.

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In the two-dimensional setting, it is known that the magnetic field can be written as the curl of the vector potential $\mathbf{A} = (0, 0, A_z)$,

$$\mathbf{B} = \nabla \times \mathbf{A}.$$

Thus, Equation (3) reduces to the following scalar equation, written in residual form:

$$R_{\mathbf{A}} = \frac{\partial A_z}{\partial t} + \mathbf{u} \cdot \nabla A_z - \frac{\eta}{\mu_0} \nabla^2 A_z + E_z^0, \tag{6}$$

where E_z^0 is some applied external electric field. The magnetic stress tensor (5) can also be rewritten in terms of A_z:

$$\mathbf{T}_M = \frac{3}{2\mu_0} \left[\frac{\partial A_z}{\partial y}\right]^2 (\hat{\mathbf{e}}_x \otimes \hat{\mathbf{e}}_x) - \frac{1}{\mu_0} \left[\frac{\partial A_z}{\partial y} \frac{\partial A_z}{\partial x}\right] (\hat{\mathbf{e}}_x \otimes \hat{\mathbf{e}}_y) - \frac{1}{\mu_0} \left[\frac{\partial A_z}{\partial y} \frac{\partial A_z}{\partial x}\right] (\hat{\mathbf{e}}_y \otimes \hat{\mathbf{e}}_x) + \frac{3}{2\mu_0} \left[\frac{\partial A_z}{\partial x}\right]^2 (\hat{\mathbf{e}}_y \otimes \hat{\mathbf{e}}_y), \tag{7}$$

where $\hat{\mathbf{e}}_x$ and $\hat{\mathbf{e}}_y$ are the unit vectors in the x and y directions, respectively. The final problem then is Equations (1), (2), and (6), with viscous and magnetic stress tensors defined by Equations (4) and (7), respectively.

1.2. **Discretization.** To discretize the problem, we use the finite-element method. We use a $Q_2 - Q_1$ (Taylor-Hood) discretization for the fluid velocity and pressure and a Q_2 discretization for the magnetic vector potential. This choice affords two advantages. First, $Q_2 - Q_1$ elements satisfy the Ladyzhenskaya-Babuska-Brezzi (LBB) condition for the Navier-Stokes part of this system. Second, discretization software was readily available for these elements [4]. We choose Q_2 elements to discretize A_z because the curl of this space a subset of a Raviart-Thomas space, a good space for $\mathbf{B}$ as we need its divergence to be bounded. When considering the components of the system separately, these appear to be acceptable choices for finite-element spaces. However, it is unclear whether these are the "best" elements to use for the combined MHD system, and further study is needed. The focus here, though, is on the solver for the resulting linear systems. After discretization and linearization from some nonlinear solver (e.g. Newton's Method), we obtain the linear system:

$$\begin{bmatrix} F & B^T & Z \\ B & 0 & 0 \\ Y & 0 & D \end{bmatrix} \begin{bmatrix} u \\ p \\ a \end{bmatrix} = \begin{bmatrix} f_u \\ f_p \\ f_a \end{bmatrix}, \tag{8}$$

where u, p, and a are the Newton corrections for $\mathbf{u}$, $\bar{p}$, and A_z, respectively, and f_u, f_p, and f_a are nonlinear residuals. We will refer to the whole system matrix as A.

2. Solving the linear system

Solution algorithms for the linear systems arising from magnetohydrodynamics models are very dependent upon the discretization as the linear systems that arise from different discretizations have very different properties. Nonlinear multigrid solvers have been used in the context of finite difference discretizations [1]. In [5], a "physics-based" preconditioner is used in this context as well, with a Krylov method (GMRES) to solve the linear systems in the implicit nonlinear solver. Nested iteration and AMG have been used with a first-order system least squares (FOSLS) approach [2]. Also, discontinuous Galerkin discretizations and their resulting solvers have been investigated [8]. Here, we investigate a finite-element discretization similar to that of [11], in which fully-coupled algebraic multilevel preconditioners are used to solve the linear systems generated by Newton linearization.

We investigate geometric multigrid preconditioners for the system (8), specifically developing effective multigrid smoothers. While we are testing these schemes as smoothers for geometric multigrid, we envision these approaches as also being applicable in an AMG context, and further study is needed. Thus, we fix everything in the multigrid method except for the smoother. First, we use geometric multigrid, coarsening by a factor of two in each direction. Second, for the grid transfers, we use the natural finite-element interpolation and its transpose for restriction. Third, we use Galerkin coarsening, defining the coarse grid operator as

$A_{2h} = RA_hP$. Finally, we use direct solves on the coarsest grid, which has 8×8 elements (948 degrees of freedom).

With the other multigrid components chosen, we turn to smoothers and consider three possibilities. The first two options can be thought of as extensions of smoothers for the incompressible fluid system to the MHD system. These are a Vanka-type smoother and a Braess-Sarazin-type smoother (see [12,13] and [3], respectively, for their introduction as smoothers for the Navier-Stokes and Stokes equations). The other option that we consider is a split preconditioner [6], where the system is "split" via an approximate block factorization into two subsystems, a fluids-magnetics system and an incompressible fluids system. The application of this smoother requires solving these two systems in sequence, to some satisfactory degree of accuracy.

3. Smoothers

3.1. **Vanka-Type Smoothers.** A general description of Vanka smoothing for incompressible fluids can be found in [10,12]. The idea of the smoother is to enforce the incompressibility constraint locally at each pressure degree of freedom (dof), updating in a Gauß-Seidel fashion. It is relatively straightforward to extend this idea to the MHD system. We index the Vanka blocks by the pressure degrees of freedom, essentially considering one row of the incompressibility constraint at each step. In effect, this means restricting the Vanka blocks to be based on overlapping 2×2 element patches. The resulting Vanka matrices are substantially smaller matrices of the same form as the system matrix A. Although each Vanka matrix defines a saddle-point problem, the construction guarantees that each is always invertible.

The Vanka smoother then iterates over all pressure degrees of freedom in a Gauß-Seidel manner and updates the degrees of freedom in the Vanka block corresponding to a pressure degree of freedom, p_ℓ, by

$$\begin{bmatrix} u \\ p \\ a \end{bmatrix}_\ell = \begin{bmatrix} u \\ p \\ a \end{bmatrix}_\ell + \begin{bmatrix} \omega_u I & & \\ & \omega_p I & \\ & & \omega_a I \end{bmatrix} \begin{bmatrix} F & B^T & Z \\ B & 0 & 0 \\ Y & 0 & D \end{bmatrix}_\ell^{-1} \begin{bmatrix} r_u \\ r_p \\ r_a \end{bmatrix}_\ell , \tag{9}$$

where r denotes the "current" residual in each component (taking into account all updates corresponding to blocks already processed, for p_l, $l < \ell$) and the ω are (optional) underrelaxation parameters. Note that this is an overlapping, multiplicative Schwarz method. As such, a degree of freedom may be updated up to four times per iteration and the order in which the updates occur is significant.

To form one of the Vanka matrices A_ℓ, we proceed analogously to the incompressible fluids case. We begin with a pressure degree of freedom and select those velocity degrees of freedom corresponding to columns with nonzero entries in the row of the incompressibility equation that corresponds to the pressure degree of freedom. This completely defines the Vanka blocks to be used for the incompressible fluids case, and the corresponding Vanka matrix would then be formed by taking the submatrix of A that is the intersection of the rows and columns corresponding to the collected degrees of freedom.

This is then extended to include the vector potential (VP) degrees of freedom. Upon inspection of the incompressibility equation, notice the lack of connections to the vector potential variable. This means that there is no implicit way to choose the VP degrees of freedom to be included in the Vanka blocks. To remedy this problem, we use those VP degrees of freedom that are collocated with those velocity degrees of freedom in which we are interested. The Vanka matrix for this pressure degree of freedom is then the submatrix of A that is the intersection of the rows and columns corresponding to the collected degrees of freedom.

Remark. It is important to note that the collocation of the velocity and VP degrees of freedom is a feature of the discretization choice. A generalization would be to choose those VP (or magnetic field, if discretized directly) degrees of freedom associated with those elements with which the pressure degree of freedom is associated.

We can also consider alternatives to the method given above for choosing the Vanka matrices in a way that leads to cheaper iterations. One possibility is the literal extension of the "diagonal Vanka" smoother [9,10] to the MHD case, where the Vanka matrices have the form:

$$A_\ell = \begin{bmatrix} \mathrm{diag}(F_\ell) & B_\ell^T & 0 \\ B_\ell & 0 & 0 \\ 0 & 0 & \mathrm{diag}(D_\ell) \end{bmatrix}.$$

This is easy to invert; however, all coupling between the velocity and vector potential degrees of freedom is entirely ignored. To avoid losing this coupling entirely, we develop an "economy Vanka" method that is still cheap but slightly more inclusive. Begin by defining the matrix H_ℓ:

$$H_\ell = \begin{bmatrix} F_\ell & Z_\ell \\ Y_\ell & D_\ell \end{bmatrix}. \tag{10}$$

Now define

$$\tilde{H}_\ell = \begin{bmatrix} \tilde{F}_\ell & \tilde{Z}_\ell \\ \tilde{Y}_\ell & \tilde{D}_\ell \end{bmatrix} \tag{11}$$

to be a new version of the matrix H_ℓ that, in each row, only contains connections to degrees of freedom located at the mesh point at which the diagonal degree of freedom is located. Thus, each row will have a maximum of three nonzeros (1 u_x dof, 1 u_y dof, and 1 A_z dof). Note that this is specific to the $Q_2 - Q_1 - Q_2$ discretization. Then we can define:

$$A_\ell = \begin{bmatrix} \tilde{F}_\ell & B_\ell^T & \tilde{Z}_\ell \\ B_\ell & 0 & 0 \\ \tilde{Y}_\ell & 0 & \tilde{D}_\ell \end{bmatrix}. \tag{12}$$

This gives a Vanka matrix that is easier to invert than the full Vanka matrix but still includes the coupling of the velocity to the vector potential.

3.2. **Braess-Sarazin Smoother.** Whereas the Vanka smoother enforces the incompressibility constraint locally, Braess-Sarazin smoothers treat this constraint globally. Braess-Sarazin smoothers were introduced for the Stokes problem in [3]. Here, we extend and apply this smoother to the MHD case. To begin, we make the following definitions:

$$\hat{F} = \begin{bmatrix} F & Z \\ Y & D \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} B & 0 \end{bmatrix}, \quad \hat{u} = \begin{bmatrix} u \\ a \end{bmatrix}, \quad \hat{f}_u = \begin{bmatrix} f_u \\ f_a \end{bmatrix}. \tag{13}$$

We then permute rows and columns of the MHD system (8) and perform the following factorization:

$$\begin{bmatrix} F & Z & B^T \\ Y & D & 0 \\ B & 0 & 0 \end{bmatrix} = \begin{bmatrix} \hat{F} & \hat{B}^T \\ \hat{B} & 0 \end{bmatrix} = \begin{bmatrix} \hat{F} & 0 \\ \hat{B} & -\hat{B}\hat{F}^{-1}\hat{B}^T \end{bmatrix} \begin{bmatrix} I & \hat{F}^{-1}\hat{B}^T \\ 0 & I \end{bmatrix}. \tag{14}$$

From this factorization of the system matrix, the Braess-Sarazin method follows by choosing an appropriate preconditioner $\bar{F}$ for $\hat{F}$ that is easy to invert and then applying the iteration

$$\begin{bmatrix} \hat{u} \\ p \end{bmatrix}^{(k+1)} = \begin{bmatrix} \hat{u} \\ p \end{bmatrix}^{(k)} + \begin{bmatrix} I & \frac{1}{\omega}\bar{F}^{-1}\hat{B}^T \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} \omega\bar{F} & 0 \\ \hat{B} & -\frac{1}{\omega}\hat{B}\bar{F}^{-1}\hat{B}^T \end{bmatrix}^{-1} \hat{r}^{(k)},$$

where $\hat{r}^{(k)}$ is the (appropriately permuted) residual at step k and ω is some damping parameter. For the incompressible fluid case, common choices for $\bar{F}$ are $\bar{F} = I$ [3] and $\bar{F} = \text{diag}(\hat{F})$ [10]. As when we considered economy Vanka above, we want to include the important coupling between the velocity and the vector potential equations (the matrices Y and Z). Thus, we follow an approach similar to (10)-(11) for economy Vanka. We define $\bar{F}$ to be the matrix $\hat{F}$, but each row now only contains those entries corresponding to degrees of freedom that are collocated with the diagonal degree of freedom. This again ensures that there are no more than three entries in each row and is specific to the $Q_2 - Q_1 - Q_2$ discretization.

3.3. **SplitPrec as a Smoother.** In [6], a split preconditioner, called SplitPrec, was proposed for the system, (8), that splits the operator into a Navier-Stokes operator and a magnetics-velocity operator, each a 2×2 block operator. SplitPrec is based on the following approximate block factorization:

$$\begin{bmatrix} F & B^T & Z \\ B & 0 & 0 \\ Y & \boxed{YF^{-1}B^T} & D \end{bmatrix} = \begin{bmatrix} F & & Z \\ & I & \\ Y & & D \end{bmatrix} \begin{bmatrix} F^{-1} & & \\ & I & \\ & & I \end{bmatrix} \begin{bmatrix} F & B^T & \\ B & 0 & \\ & & I \end{bmatrix}. \tag{15}$$

The highlighted term is the error that is introduced by making this approximate factorization.

The application of this method as a smoother is an approximate solve with the fluid-magnetics system (hereafter, we will refer to this as the "magnetics system" as the other system involves no magnetics coupling) followed by an approximate solve with the fluids-only system (hereafter, we will refer to this as the "fluids system"):

$$
\begin{aligned}
r^{(k)} &= b - Ax^{(k)} \\
\hat{r} &= \begin{bmatrix} F & 0 & Z \\ 0 & I & 0 \\ Y & 0 & D \end{bmatrix}^{-1} r^{(k)} \\
r^{*} &= \begin{bmatrix} F & B^T & 0 \\ B & 0 & 0 \\ 0 & 0 & I \end{bmatrix}^{-1} (r - A\hat{r}) \\
x^{(k+1)} &= x^{(k)} + \hat{r} + r^{*}
\end{aligned}
$$

We then use previously developed and well-known techniques for the incompressible fluids system (Braess-Sarazin, Vanka, or even V-Cycles with either of these as the smoother). We also need to approximately solve the magnetics system. Based on preliminary numerical experiments (summarized below), it seems that point Gauß-Seidel and block Gauß-Seidel with relatively simple blocks are easy and effective options.

4. Test Problem and Results

4.1. The Hartmann Problem. For numerical experiments, we study a modified Hartmann flow vector potential problem [11, §5.1.2]. This is a steady-state problem posed over a square box $(x, y) \in [-1, 1]^2$ with a pressure gradient that drives the flow $\frac{\partial \bar{p}}{\partial x} = -G_0$. The velocity and magnetic fields have solutions $\mathbf{u} = (u_x, 0, 0)$ and $\mathbf{B} = (B_x, B_0, 0)$, where B_0 is an applied external magnetic field. The other components are

$$
\begin{aligned}
u_x &= -\frac{\rho G_0 Ha}{\mu_0 B_0^2}\left[\frac{\cosh(Ha) - \cosh(yHa/L)}{\sinh(Ha)}\right] \\
B_x &= -\frac{B_0 Re_m}{Ha}\left[\frac{\sinh(yHa/L) - (y/L)\sinh(Ha)}{\cosh(Ha) - 1}\right].
\end{aligned}
$$

In terms of the vector potential A_z, this solution becomes

$$
A_z = -B_0 x - \frac{B_0 Re_m}{Ha}\left[\frac{\cosh(yHa)/Ha - [y^2/(2L)]\sinh(Ha)}{\cosh(Ha) - 1}\right].
$$

An external electric field is needed to sustain this solution and is given by

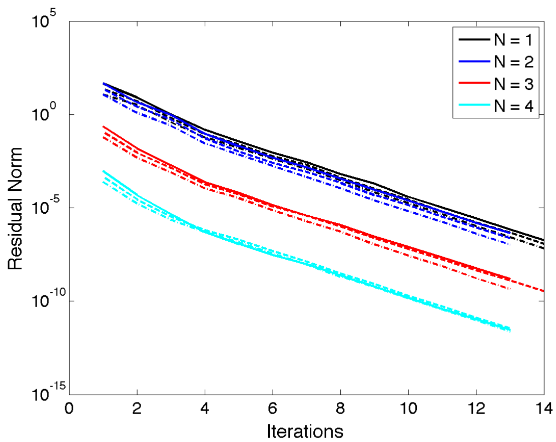
$$
E_z^0 = \frac{G_0}{B_0}[Ha\coth(Ha) - 1].
$$

Here, the Reynolds and Lundquist numbers are defined by $Re = 2U/\nu$, $Re_m = 2\mu_0 U/\eta$, respectively, where U is the maximum x-direction velocity. The Hartmann number, Ha, is defined as $Ha = 2B_0/\sqrt{\rho\nu\eta}$. For our purposes here, we have taken $\rho = \nu = \eta = \mu_0 = 1$ and selected different values for G_0 and B_0 to produce the desired Reynolds, Lundquist, and Hartmann numbers.

4.2. Numerical Experiments. For each smoother described above, we test a variety of parameters. Physically, we vary the Hartmann number, testing values $Ha = 1, 5, 20$. In effect, this tests applied magnetic fields of different strengths and, by extension, different velocity profiles. Numerically, we vary both grid sizes and the properties of the linear systems using a series of reference linearizations, generated by applying Newton's method with an unrelated solver. This ensures that we are comparing the effectiveness of these solvers on a consistent set of linear systems; further study is needed on the effects of these linear solvers on nonlinear convergence.

In the following plots, the various colors indicate the reference Newton linearization, as shown in the legend. The type of line indicates grid size: solid lines (−) for 32×32 elements, dashed lines (−−) for 64×64 elements, and dash-dotted lines (−·) for 128×128 elements. In general, the behavior across Hartmann numbers did not differ significantly, and, therefore, we only show results for one Hartmann number, namely $Ha = 5$. In such cases that there was significant variation, multiple values of Ha are shown. As described above, the solver that we apply in each case is multigrid-preconditioned GMRES. Specifically, we use a V(1,1) cycle as the preconditioner and choose the multigrid smoother to be one of the three described above.

4.2.1. *Vanka Results.* We apply the solver using full Vanka smoothing for the multigrid smoother. For the underrelaxation parameters, we choose $\omega_u = 0.8$, $\omega_p = 0.6$, and $\omega_a = 0.6$. These values are based on a numerical study to determine, in steps of 0.1, the best combination of underrelaxation parameters. This combination is used here because it yields good results across all test problems used in that study. Further study of this choice is, of course, needed. The results for this experiment are shown on the left in Figure 1. The figure shows that this smoother leads to consistently good performance across all grid sizes and reference linearizations.



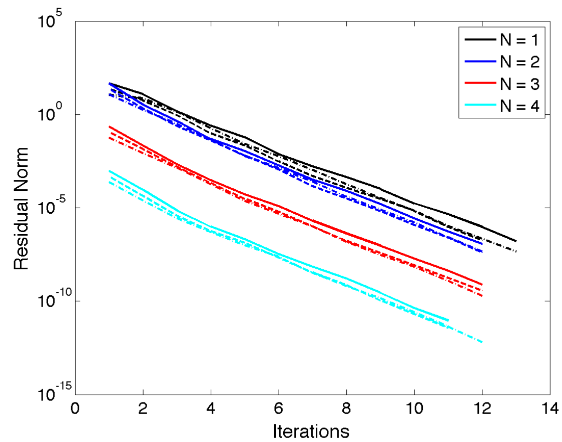


FIGURE 1. Convergence of MG-preconditioned GMRES on the $Ha = 5$ test problem using full Vanka smoothing on the left and economy Vanka smoothing on the right with underrelaxation parameters $\omega_u = 0.8$, $\omega_p = 0.6$, and $\omega_a = 0.6$.

We also apply the solver using economy Vanka smoothing for the multigrid smoother. The underrelaxation parameters here are taken to be the same as for the full Vanka smoothing. The results for this experiment are shown on the right in Figure 1. Again, we see that this method yields scalable performance for all reference linearizations.

4.2.2. *Braess-Sarazin Results.* Next, we apply the solver using Braess-Sarazin for the multigrid smoother. The smoothing parameter ω is taken in each case to be the spectral radius of $\bar{F}^{-1}\hat{F}$, as suggested by [10]. This is computed using MATLAB's `eigs` function. To solve the approximate Schur complement equation required in this iteration, we use three sweeps of symmetric Gauß-Seidel. The results are shown in Figure 2. We see that this method also shows scalable performance across all reference linearizations.

4.2.3. *SplitPrec Results.* Finally, we apply the solver using SplitPrec as the multigrid smoother, with several combinations of solvers for the fluids and magnetics systems. We need to determine if the method is viable at all, and thus we begin by testing SplitPrec using direct solves for both systems. The results are shown in Figure 3. These results show that SplitPrec is generally an effective smoother, although some lack of scalability is seen.

Having determined that SplitPrec can, in fact, be used effectively in this context using direct solves, it remains to be seen if this approach is viable if we do not use direct solves on both of the subsystem solves. First, we remove the direct solve on the magnetics subsystem and replace it with three sweeps of symmetric Gauß-Seidel. The results are shown in Figure 4. While this variant generally performs well, it is unable to effectively solve the first reference linearization at higher Hartmann numbers on the coarsest mesh.

Next, we switch to approximate solves for both systems. Thus, we use three sweeps of symmetric Gauß-Seidel for the magnetics system and two iterations of Braess-Sarazin for the fluids system. The results are shown in Figure 5. We use two V(2,2) cycles with symmetric Gauß-Seidel as the smoother to solve the

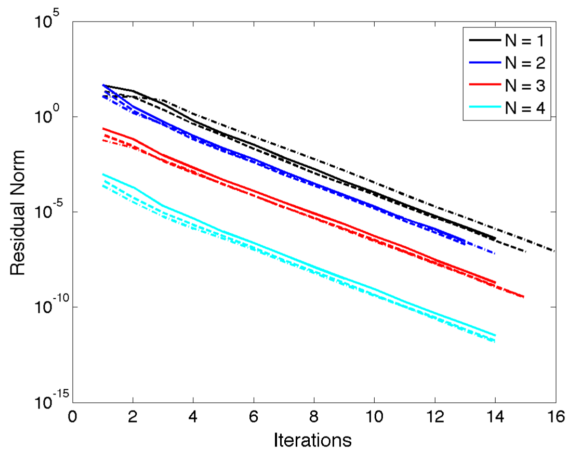


FIGURE 2. Convergence of MG-preconditioned GMRES on the $Ha = 5$ test problem using Braess-Sarazin smoothing.

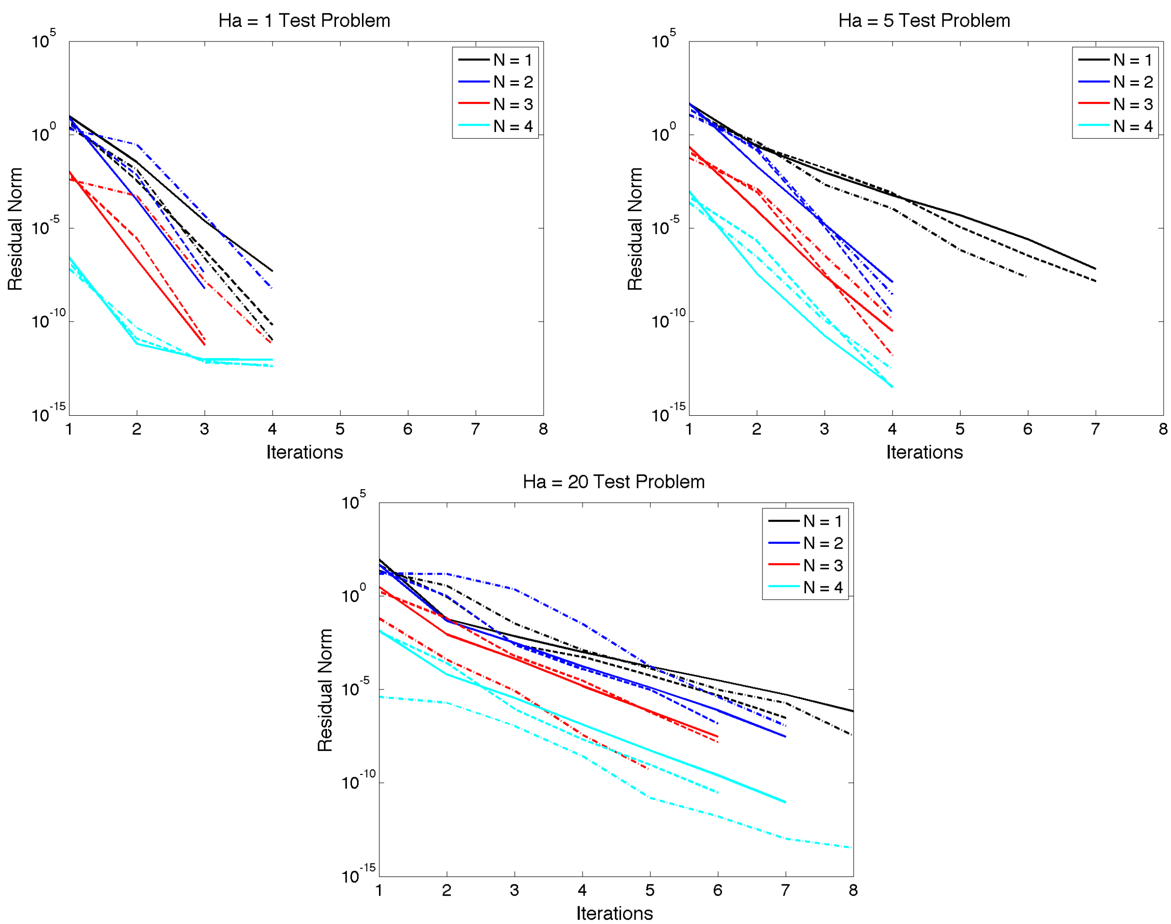


FIGURE 3. Convergence of MG-preconditioned GMRES on all test problems using SplitPrec as the multigrid smoother, with direct solves for both the magnetics and the fluids system solves.

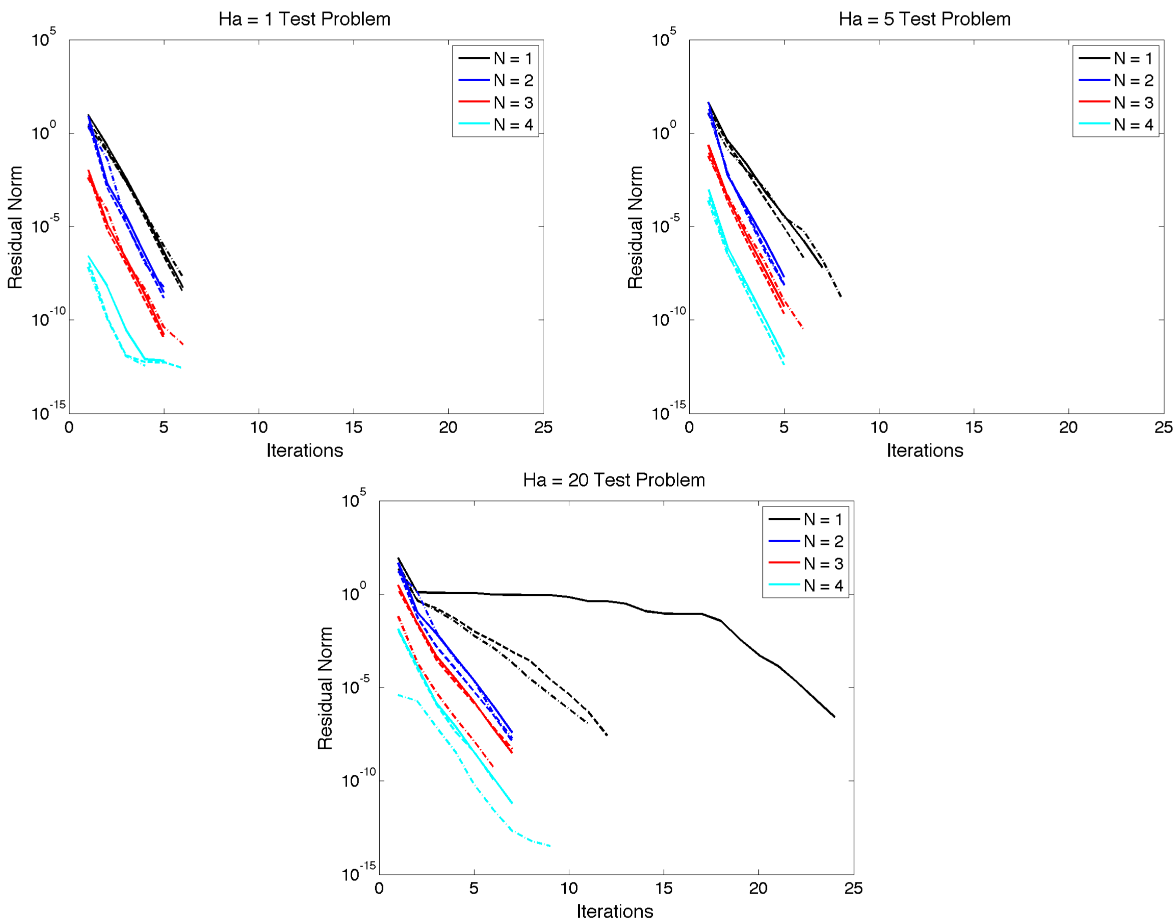


Figure 4. Convergence of MG-preconditioned GMRES on all test problems using SplitPrec as the multigrid smoother, with three sweeps of symmetric Gauß-Seidel to approximately solve the magnetics system and a direct solve for fluids system solve.

approximate Schur complement system in the Braess-Sarazin iteration. In this case, we generally see good performance for the smaller grid sizes, but lack convergence on the 128×128 grid (a second V(2,2) is required to get those problems to converge at all in fewer than 50 iterations).

Since Braess-Sarazin did not seem to be effective, we then tried two sweeps of Vanka smoothing. The results are shown in Figure 6. We used $\omega_u = \omega_p = 0.7$ for the underrelaxation parameters. These results show that this version of the smoother is highly effective and scalable across all reference linearizations, with the notable exception, again, of the first reference linearization for the problem $Ha = 20$. Thus it is clear that effective smoothers for the incompressible fluids system do not necessarily lead to effective MHD smoothers within SplitPrec.

5. Conclusions and Future Directions

We have presented three potential smoothers for a vector-potential incompressible resistive MHD formulation. The Vanka and Braess-Sarazin smoothers come from extensions to smoothers for incompressible fluids systems, while the SplitPrec smoother attempts to split the physics into two simpler systems that are then solved independently. For the test problem that we consider, we see that the Vanka and Braess-Sarazin methods consistently perform and scale well. We also observed that the SplitPrec smoother could perform well, but it was difficult to find combinations of solvers for the subsystems that yield as good of performance.

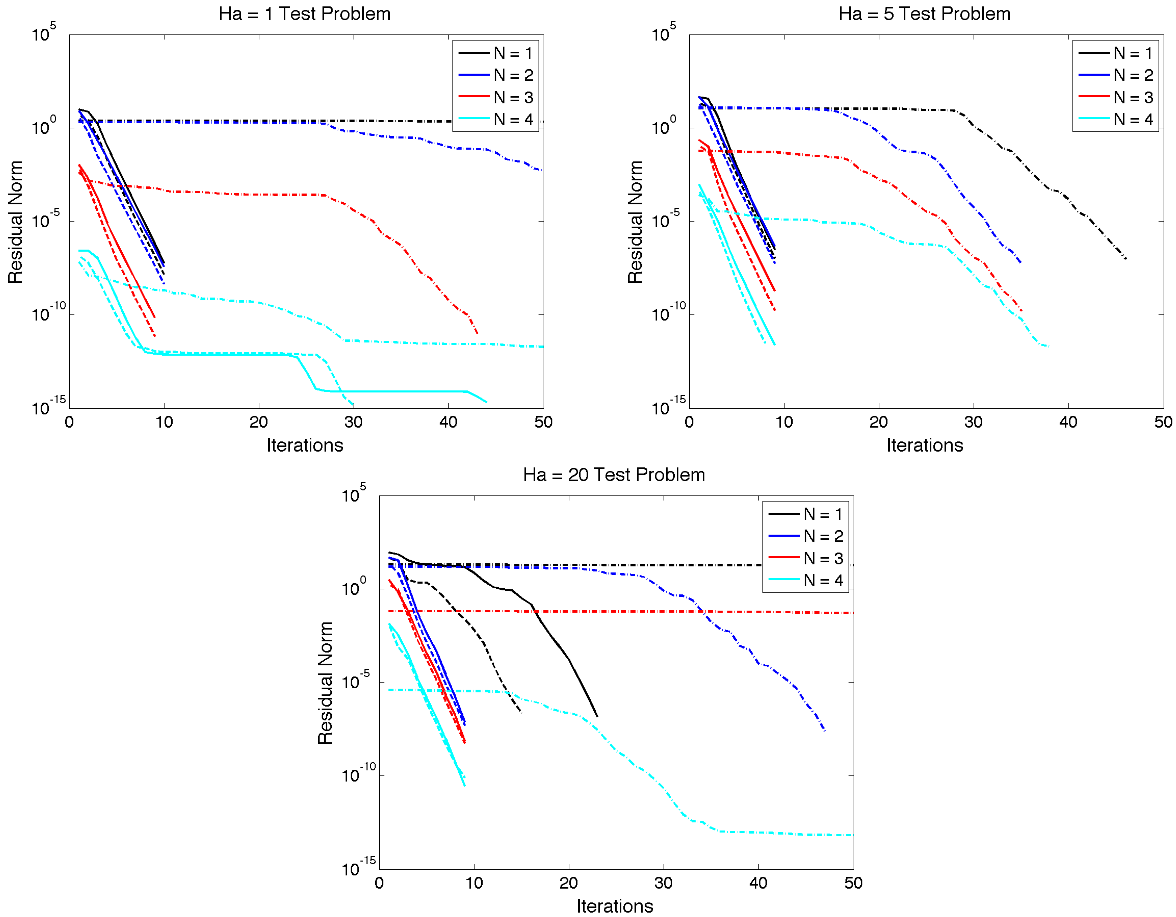


FIGURE 5. Convergence of MG-preconditioned GMRES on all test problems using SplitPrec as the multigrid smoother, with three sweeps of symmetric Gauß-Seidel to solve the magnetics system and two iterations of Braess-Sarazin for the fluids system solve.

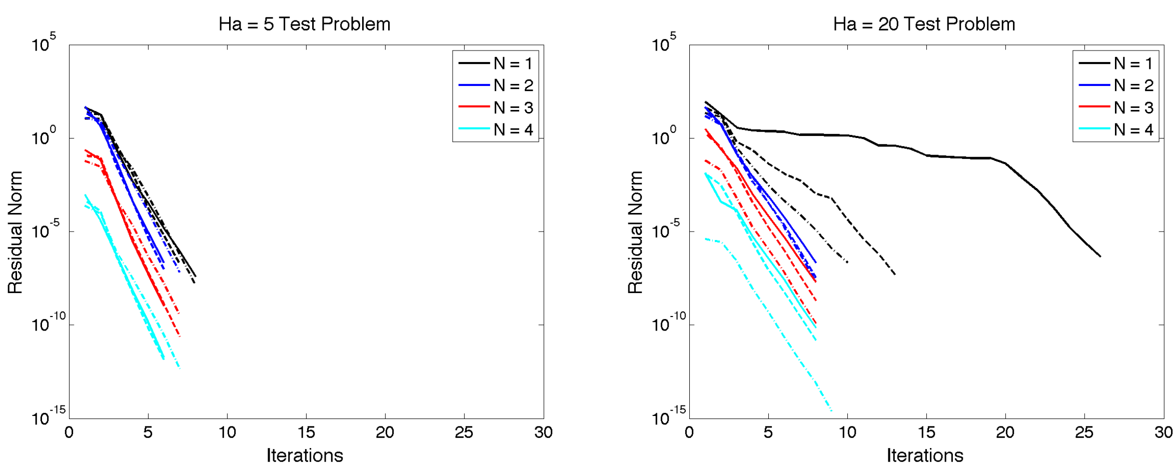


FIGURE 6. Convergence of MG-preconditioned GMRES on the $Ha = 5$ (left) and $Ha = 20$ (right) test problems using SplitPrec as the multigrid smoother. We used three sweeps of symmetric Gauß-Seidel to solve the magnetics system and a two iterations of Vanka smoothing for the fluids system solve.

Up to this point, all of the numerical experiments have been run in MATLAB and we have been unable to effectively compare these smoothers in terms of computation time. Thus, we are moving to a C++ implementation using the Trilinos framework from Sandia National Laboratories in order to gather meaningful timing results. This will allow us to determine which of the above smoothers are the most efficient.

Finally, it is unclear that the $Q_2 - Q_1 - Q_2$ discretization is the optimal choice for the combined MHD problem. Thus, we will test these smoothers on other problems using other discretizations such as those that preserve the physical quantities involved. We will also consider directly discretizing the magnetic field, $\mathbf{B}$, rather than considering the vector potential formulation. This requires considering methods that preserve the solenoidal constraint, $\nabla \cdot B = 0$. Finally, we would like to study the various smoothers in the context of the full nonlinear solver and investigate how to improve the cost of solving the full nonlinear system efficiently.

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