# SYMMETRIC MULTIGRID THEORY FOR DEFLATION METHODS

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**Abstract.** We derive an estimate for the speed of convergence of deflation methods, based on the idea of Nicolaides [9], for the iterative solution of linear systems of equations. This is done by using results from classical algebraic multigrid theory. As a further result we obtain that many prolongation operators from multigrid methods can be used to span the deflation subspace, which is needed for deflation methods.

**Key words.** conjugate gradients; deflation; algebraic multigrid; convergence estimates

## 1. Preliminaries.

Consider solving the linear system of equations

$$Ax = b\,, \tag{1.1}$$

where $A \in \mathbb{K}^{n\times n}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$) is self-adjoint and positive definite and $x, b \in \mathbb{K}^n$. Furthermore we are interested in the case where the matrix $A$ is large and sparse. The conjugate gradient (cg) method [7, 11] is an iterative method which is often well suited to solve these systems. Deflation methods have been successfully used to improve the speed of convergence of the cg method [6, 9, 12]. These methods need the construction of a *deflation subspace*. In this paper we show that many prolongation operators from multigrid methods can be used to construct the deflation subspace and derive a convergence estimate for this method. By doing so we are able show that with the right choice of a deflation subspace, deflation methods have convergence properties similar to multigrid methods without the need of constructing an appropriate smoother.

The paper is structured as follows. The rest of this section introduces some basic notation. Section 2 gives a short introduction into deflation methods. In section 3 we analyse the convergence of deflation methods by analysing the condition of the matrix $A(I - \pi_A(V)) \in \mathbb{K}^{n\times n}$. Finally, we use some results from multigrid theory to derive a lower bound for the smallest eigenvalue of the matrix in section 4 and 5. Section 6 shows how the prolongation operators from the classical algebraic multigrid method for $M$-matrices can be used to obtain the deflation subspace. In section 7 we present some numerical experiments confirming the theory.

Assume that $\tilde{x} \in \mathbb{K}^n$ is an approximation to $x$. Then the residuum $r \in \mathbb{K}^n$ is given by $r = b - A\tilde{x}$, the error $e = x - \tilde{e}$. Note that

$$Ae = r\,.$$

Since $A$ is self-adjoint and positive definite we can define by

$$\langle x, y\rangle_A := \langle Ax, y\rangle \quad \text{and} \quad \|x\|_A := \sqrt{\langle x, x\rangle_A}$$

the $A$-inner product and the $A$-norm, respectively. We define $\mathbb{K}_k[X]$ as the set of polynomials in the variable $X$ whose degree is less than or equal to $k$. Let $\mathcal{S} \subseteq \mathbb{K}^n$ be a subspace then $\mathcal{S}^\perp$ is its orthogonal complement with respect to the 2-inner product and $\mathcal{S}^{\perp_A}$ is its orthogonal complement with respect to the $A$-inner product.

## 2. Deflation Methods.

Deflation methods are often used to speed up the convergence of Krylov subspace methods such as the cg method. In this section we briefly review the main idea of these methods.

The $m^{\text{th}}$ Krylov subspace $\mathcal{K}_m(A, v)$ corresponding to a matrix $A \in \mathbb{K}^{n\times n}$ and a vector $v \in \mathbb{K}^n$ is given by

$$\mathcal{K}_m(A, v) := \operatorname{span}\{v, Av, A^2v, \ldots, A^{m-1}v\} = \{P(A)v : P \in \mathbb{K}_{m-1}[X]\}\,.$$

The cg method is typically used as an iterative method, which generates the iterates $x_1, x_2, x_3, \ldots \in \mathbb{K}^n$ for a given initial guess $x_0 \in \mathbb{K}^n$, where

$$x_i = x_0 + \tilde{e}_i \quad \text{and} \quad \tilde{e}_i \in \mathcal{K}_i(A, r_0)$$

such that the error $e_i = x - x_i$ fulfills

$$\|e_i\|_A = \|x - x_i\|_A = \min_{\tilde{x}\in x_0+\mathcal{K}_i} \|x - \tilde{x}\|_A\,.$$

Note that since $e_i = x - x_i = x - (x_0 + \tilde{e}_i) = e_0 - \tilde{e}_i$ especially $e_0 = x - x_0$ and $\tilde{e} = \tilde{x} - x_0$ this is equivalent to

$$\|e_i\|_A = \|e_0 - \tilde{e}_i\|_A = \min_{\tilde{e}\in\mathcal{K}_i} \|e_0 - \tilde{e}\|_A\,.$$

By definition of the Krylov subspace and since $\tilde{e}_i \in \mathcal{K}_i$ we can write

$$\tilde{e}_i = \tilde{P}(A)r_0 = \tilde{P}(A)Ae_0$$

with $\tilde{P} \in \mathbb{K}_{i-1}[X]$ and thus

$$e_i = e_0 - \tilde{e}_i = e_0 - \tilde{P}(A)Ae_0 = P(A)e_0 \quad \text{where} \quad P = 1 - X\tilde{P}\,. \tag{2.1}$$

A polynomial $P$ can be written in the form (2.1) if and only if $P \in \mathbb{K}_i[X]$ and $P(0) = 1$. Hence

$$\|e_i\|_A = \min_{\substack{P\in\mathbb{K}_i[X]\\ P(0)=1}} \|P(A)e_0\|_A\,. \tag{2.2}$$

Recall that the matrix $A$ is self-adjoint positive definite. Hence there exists a unitary matrix $Q \in \mathbb{K}^{n\times n}$, i.e. $Q^* = Q^{-1}$, and a corresponding diagonal matrix $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n) \in \mathbb{K}^{n\times n}$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ such that $A = Q\Lambda Q^*$. Then the columns $q_1, \ldots, q_n \in \mathbb{K}^n$ of $Q$ form a basis of eigenvectors corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_n$.

Writing the initial error $e_0$ as $e_0 = \xi_1 q_1 + \ldots + \xi_n q_n$ and using $P(A)q_i = P(\lambda_i)q_i$ equation (2.2) yields

$$\|e_i\|_A^2 = \min_{\substack{P\in\mathbb{K}_i[X]\\ P(0)=1}} \|\sum_{i=1}^{n} \xi_i P(A)e_0\|_A^2 = \min_{\substack{P\in\mathbb{K}_i[X]\\ P(0)=1}} \sum_{i=1}^{n} |\xi_i|^2 |P(\lambda_i)|^2 \lambda_i\,. \tag{2.3}$$

Thus the cg method constructs a polynomial such that the point $(0, 1)$ is interpolated and the points $(\lambda_i, 0)$ are approximated. The approximation at $\lambda_i$ is weighted by $|\xi_i|^2\lambda_i$. That means that the values $|\xi_i|^2\lambda_i$ represent the constraints applied to the minimization. Weakening these constraints should increase the speed of convergence. It would be desirable to set $\xi_i = 0$ for some $i$. However this is impractical since the exact eigenvectors are usually not known. In addition due to roundoff errors $\xi_i$ may become nonzero during the algorithm. Fortunately it is sufficient to make $\xi_i$ "small

enough" for some $i$, such that the cg method to computes an approximation to the solution which is good enough, before the weakened constraint is "considered" in the minimization process.

The main idea of a deflation method is to split the error into two components. Assume that we are given a subspace $\mathcal{S} \subset \mathbb{K}^n$ which contains the error components for the matrix $A$ that are responsible for slow convergence, e.g. a space spanned by some eigenvectors corresponding to the smallest eigenvalues of $A$. Let the columns of $V \in \mathbb{K}^{n\times m}$ be a basis of $\mathcal{S}$. From now on let the initial guess be given by $x_{-1}$ and the initial error by $e_{-1}$. We want to compute a vector $\tilde{e}_0$ that removes the troublesome components from $e_{-1}$. Thus we set $x_0 = x_{-1} + \tilde{e}_0$ and obtain $e_0 = e_{-1} - \tilde{e}_0$. Since

$$\|e_{-1} - \tilde{e}_0\|_A^2 = \|e_0\|_A^2 = \sum_{i=1}^{n} |\xi_i|^2 \lambda_i$$

is the sum over all weights, we want to choose $\tilde{e}_0 \in \mathcal{S}$ such that $\|e_{-1} - \tilde{e}_0\|_A$ is minimized. It is clear that $\tilde{e}_0$ is the $A$-orthogonal projection of $e_0$ onto $\mathcal{S}$ which we denote by $\pi_A(V)e_{-1}$. As

$$\tilde{e}_0 = \pi_A(V)e_{-1} = V(V^*AV)^{-1}V^*Ae_{-1} = V(V^*AV)^{-1}V^*r_{-1}$$

we can compute $\tilde{e}_0$ without knowledge of $e_{-1}$. Additionally from

$$e_{-1} = \pi_A(V)e_{-1} + (I - \pi_A(V))e_{-1} = \tilde{e}_0 + e_0$$

and $x = x_{-1} + e_{-1}$ we can compute the solution if we can compute $(I - \pi_A(V))e_{-1}$ which is the $A$-orthogonal projection of $e_{-1}$ onto $\mathcal{S}^{\perp_A}$. Modifying the cg method to restrict the search directions to the subspace $\mathcal{S}^{\perp_A}$ and thus minimizing the $A$-norm of the error over the subspace $\mathcal{S}^{\perp_A}$ yields a method [9, 12] which computes the desired projection.

Another viable approach is to solve

$$A(I - \pi_A(V))\, z = r_0 \tag{2.4}$$

and obtain the solution $x = x_0 + (I - \pi_A(V))z$. This is indeed the solution because $Ae_0 = r_0$ and $e_0 \in \text{range}(I - \pi_A(V))$ and thus $e_0 = (I - \pi_A(V))z$. Note that the matrix of (2.4) is singular. Using the relation

$$A(I - \pi_A(V)) = A - AV(V^*AV)^{-1}V^*A = (I - \pi_A(V))^*A$$

and the fact that $(I - \pi_A(V))$ is a projection yields

$$A(I - \pi_A(V)) = A(I - \pi_A(V))(I - \pi_A(V)) = (I - \pi_A(V))^*A(I - \pi_A(V))\,. \tag{2.5}$$

That is the matrix $A(I - \pi_A(V))$ is self-adjoint positive semi-definite. In [8] it has been shown that the cg method can be used to solve singular systems as well. Hence we can apply the method to solve (2.4). In case we choose the initial guess $x_{-1} = 0$ we obtain the method presented in [6].

Since $A(I - \pi_A(V))$ is self-adjoint positive semi-definite there exists a unitary matrix $U$ and a diagonal matrix $D = \text{diag}(\mu_1, \ldots, \mu_n)$ such that $A(I - \pi_A(V)) = UDU^*$ and $\mu_1 \geq \cdots \geq \mu_n \geq 0$. The values $\mu_1, \ldots, \mu_n$ are eigenvalues of $A(I - \pi_A(V))$. Let $\ell \in \mathbb{N}$ be the largest index such that $\mu_\ell \neq 0$. The speed of convergence can be estimated by

$$\|e_i\|_A \leq 2\left(\frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1}\right)^i \|e_0\|_A \quad \text{for } i = 0, 1, 2, \ldots$$

in both methods, where $\kappa = \frac{\mu_1}{\mu_\ell}$ [6, 12]. Hence we are interested in estimating the largest and smallest nonzero eigenvalue of the matrix $A(I - \pi_A(V))$.

**3. Eigenvalue Bounds.** In this section we derive an upper bound for the largest eigenvalue and a representation for the smallest nonzero eigenvalue of $A(I - \pi_A(V))$. First of all note that due to the properties of the Rayleigh quotient we have

$$\mu_1 = \max_{x \in \mathbb{K}^n \setminus \{0\}} \frac{\langle A(I - \pi_A(V))x, x\rangle}{\langle x, x\rangle} .$$

In addition from the fact that $A\pi_A(V) = AV(V^*AV)^{-1}V^*A$ is positive semi-definite and thus

$$\langle A(I - \pi_A(V))x, x\rangle = \langle Ax, x\rangle - \langle A\pi_A(V)x, x\rangle \leq \langle Ax, x\rangle ,$$

we obtain

$$\mu_1 \leq \max_{x \in \mathbb{K}^n \setminus \{0\}} \frac{\langle Ax, x\rangle}{\langle x, x\rangle} = \lambda_1 = \|A\| . \tag{3.1}$$

This gives a trivial upper bound for the largest eigenvalue. The following theorem will later be used to derive a lower bound for the smallest eigenvalue. This is just a simplification of the well known min-max (or Courant-Weyl) theorem.

THEOREM 3.1. *Let $M \in \mathbb{K}^{n\times n}$ be self-adjoint. There exists a unitary matrix $U \in \mathbb{K}^{n\times n}$ and a diagonal matrix $D = \operatorname{diag}(\mu_1, \ldots, \mu_n)$ such that $M = UDU^*$ and $\mu_1 \geq \ldots \geq \mu_n$. The columns $u_1, \ldots, u_n$ of $U$ are the eigenvectors corresponding to the eigenvalues $\mu_1, \ldots, \mu_n$. Then*

$$\mu_k = \min_{\substack{x \in \mathbb{K}^n \setminus \{0\} \\ x \perp \operatorname{span}\{u_{k+1}, \ldots, u_n\}}} \frac{\langle Mx, x\rangle}{\langle x, x\rangle} .$$

*Proof.* Let $\mathfrak{e}_i$ be the $i^{\text{th}}$ unit vector. Let $x \in \mathbb{K}^n \setminus \{0\}$ and $x \perp \operatorname{span}\{u_{k+1}, \ldots, u_n\}$. We have

$$\langle Mx, x\rangle = \langle U^*DUx, x\rangle = \langle DU^*x, U^*x\rangle = \langle Dz, z\rangle$$

with $z = U^*x$. Since $x \perp \operatorname{span}\{u_{k+1}, \ldots, u_n\}$ we have $z_{k+1} = z_{k+2} = \cdots = z_n = 0$. Thus

$$\begin{aligned} \langle Dz, z\rangle &= \langle \mu_1 z_1 \mathfrak{e}_1 + \ldots + \mu_k z_k \mathfrak{e}_k, z\rangle = \mu_1 \langle z_1 \mathfrak{e}_1, z\rangle + \ldots + \mu_k \langle z_k \mathfrak{e}_k, z\rangle \\ &\geq \mu_k \langle z_1 \mathfrak{e}_1, z\rangle + \ldots + \mu_k \langle z_k \mathfrak{e}_k, z\rangle = \mu_k \langle z, z\rangle , \end{aligned}$$

which yields

$$\langle Mx, x\rangle \geq \mu_k \langle z, z\rangle = \mu_k \langle x, x\rangle .$$

The assertion follows from

$$\langle Mu_k, u_k\rangle = \langle \mu_k u_k, u_k\rangle = \mu_k \langle u_k, u_k\rangle . \qquad \square$$

To find the smallest nonzero eigenvalue $\mu_\ell$ of $A(I - \pi_A(V))$ we need to determine $\operatorname{span}\{u_{\ell+1}, \ldots, u_n\}$. Since these are the eigenvectors corresponding to the eigenvalue zero their span is the kernel of the matrix. Since the matrix $A$ has full rank the kernel

of $A(I-\pi_A(V))$ is the kernel of $(I-\pi_A(V))$. The kernel is the span of the columns of $V$ which we denoted by $\mathcal{S}$. Thus

$$\mu_\ell = \min_{x\in\mathcal{S}^\perp\setminus\{0\}} \frac{\langle A(I-\pi_A(V))x, x\rangle}{\langle x, x\rangle}$$

and due to (2.5)

$$\begin{aligned}\mu_\ell &= \min_{x\in\mathcal{S}^\perp\setminus\{0\}} \frac{\langle (I-\pi_A(V))^* A(I-\pi_A(V))x, x\rangle}{\langle x, x\rangle} \\ &= \min_{x\in\mathcal{S}^\perp\setminus\{0\}} \frac{\langle A(I-\pi_A(V))x, (I-\pi_A(V))x\rangle}{\langle x, x\rangle} \\ &= \min_{x\in\mathcal{S}^\perp\setminus\{0\}} \frac{\|(I-\pi_A(V))x\|_A^2}{\|x\|_2^2}\,. \end{aligned} \tag{3.2}$$

**4. Weak Approximation Property.** In this section we use some results from the algebraic multigrid theory to obtain an estimate for $\mu_\ell$.

Algebraic multigrid methods [13] are based on the assumption that the error can be split into highly oscillatory and slowly varying components. The smoother reduces the highly oscillatory components while the coarse grid correction reduces the slowly varying ones. In many application the highly oscillatory components are spanned by the eigenvectors corresponding to large eigenvalues while the slowly varying components are spanned by the eigenvalues corresponding to the small eigenvalues of the matrix. In order to quantify those properties classical algebraic multigrid theory [2] measures how well the coarse grid correction, defined by the *prolongation operator*, is able to reduce the slowly varying error components. Due to the connection to the eigenvectors corresponding to the small eigenvalues, we expect this theory to give a lower bound for the eigenvalues of $A(I-\pi_A(V))$.

We introduce and use the *weak approximation property* which can be found in a slightly modified version in [3] and is a weaker requirement than the one formulated in [2], as we will see in section 6.

DEFINITION 4.1. *An operator $V\in\mathbb{K}^{n\times m}$, $m\le n$, fulfills the* weak approximation property *if there exists a function $R:\mathbb{K}^n\to\mathbb{K}^m$ and a number $K\ge 0$ such that*

$$\|x - VR(x)\|_2^2 \le \frac{K}{\|A\|}\|x\|_A^2 \quad \textit{for all } x\in\mathbb{K}^n\,. \tag{4.1}$$

Using this definition the following theorem gives an estimate for the smallest eigenvalue of the matrix $A(I-\pi_A(V))$.

THEOREM 4.2. *Let $V\in\mathbb{K}^{n\times m}$ be a matrix such that $V$ has full rank and the weak approximation property (4.1) is fulfilled. Then we can estimate the condition $\kappa=\frac{\mu_1}{\mu_\ell}$ of the matrix $A(I-\pi_A(V))$ by*

$$\kappa \le \frac{K}{\xi} \qquad \textit{where} \qquad \xi := \min_{x\in\mathcal{S}^\perp\setminus\{0\}} \frac{\|x-\pi_A(V)x\|_A^2}{\|x\|_A^2}\,. \tag{4.2}$$

*Proof.* Let $\mathcal{S}\subseteq\mathbb{K}^n$ be the subspace spanned by the columns of $V$ and $\pi(V)$ be the orthogonal projection onto $\mathcal{S}$ with respect to the 2-inner product. Then

$$\|x-\pi(V)x\|_2^2 \le \|x-y\|_2^2 \quad \text{for all } y\in\mathcal{S}\,. \tag{4.3}$$

For $x \in \mathcal{S}^\perp$ we have $\pi(V)x = 0$ and thus due to (4.3) and (4.1)

$$\|x\|_2^2 = \|x - \pi(V))x\|_2^2 \leq \|x - VR(x)\|_2^2 \leq \frac{K}{\|A\|}\|x\|_A^2 \quad \text{for } x \in \mathcal{S}^\perp\,.$$

Furthermore due to (3.2) we obtain

$$\mu_\ell \geq \frac{\|A\|}{K} \min_{x \in \mathcal{S}^\perp \setminus \{0\}} \frac{\|x - VR(x)\|_A^2}{\|x\|_A^2}\,. \tag{4.4}$$

The property of the $A$-orthogonal projection that

$$\|x - \pi_A(V)x\|_A^2 \leq \|x - y\|_A^2 \quad \text{for all } x \in \mathcal{S}$$

yields $\|x - \pi_A(V)\|_A^2 \leq \|x - VR(x)\|_A^2$. Thus we obtain due to (4.4) that

$$\mu_\ell \geq \frac{\|A\|}{K} \min_{x \in \mathcal{S}^\perp \setminus \{0\}} \frac{\|x - \pi_A(V)x\|_A^2}{\|x\|_A^2} = \frac{\|A\|}{K}\xi\,. \tag{4.5}$$

Hence we obtain by (3.1) and (4.5)

$$\kappa = \frac{\mu_1}{\mu_\ell} \leq \frac{\|A\|}{\frac{\|A\|}{K}\xi} = \frac{K}{\xi}\,. \qquad \square$$

**5. Strengthened Cauchy-Schwarz Inequality.** From the above section we see that to estimate the condition number $\kappa$ we need to find an estimate for $\xi$. As we show in this section, this can be estimated by the *strengthened Cauchy-Schwarz inequality*. To do so, we use theory developed in [1] and [5].

For two subspaces $H_1, H_2 \subseteq \mathbb{K}^n$ with $H_1 \cap H_2 = \{0\}$ there exists a constant $\gamma \in [0, 1)$ such that

$$|\langle u, v\rangle| \leq \gamma\sqrt{\langle u, u\rangle}\sqrt{\langle v, v\rangle} \quad \forall u \in H_1,\ \forall v \in H_2 \tag{5.1}$$

[4, theorem 2.1]. Equation (5.1) is called *strengthened Cauchy-Schwarz inequality* and $\gamma$ can be interpreted as the *abstract angle* between $H_1$ and $H_2$.

Let us assume that (5.1) is fulfilled and $u \in H_1$, $v \in H_2$. Since we have in general $|a_1 a_2| \leq \frac{1}{2}(a_1^2 + a_2^2)$ for $a_1, a_2 \in \mathbb{R}$

$$|\langle u, v\rangle| \leq \gamma\,\langle u, u\rangle^{\frac{1}{2}}\,\langle v, v\rangle^{\frac{1}{2}} \leq \tfrac{\gamma}{2}\left[\langle u, u\rangle + \langle v, v\rangle\right]\,.$$

Hence

$$\begin{aligned}(1-\gamma)\left[\langle u, u\rangle + \langle v, v\rangle\right] &\leq \left[\langle u, u\rangle + \langle v, v\rangle\right] - \gamma\left[\langle u, u\rangle + \langle v, v\rangle\right] \\ &\leq \left[\langle u, u\rangle + \langle v, v\rangle\right] - 2\,|\langle u, v\rangle| \\ &\leq \left[\langle u, u\rangle + \langle v, v\rangle\right] - 2\,\mathrm{Re}\,\langle u, v\rangle \\ &= \langle u + v, u + v\rangle\,.\end{aligned}$$

Taking the infimum over all $v \in H_2$ yields

$$(1-\gamma)\|u\|^2 = \inf_{v \in H_2}\|u + v\|^2 \quad \forall u \in H_1\,. \tag{5.2}$$

This general result is applied in the case where $H_1 = \mathcal{S}^\perp$, $H_2 = \mathcal{S}$ and $\langle\cdot,\cdot\rangle$ is the $A$-inner product. The $A$-orthogonal projection $\pi_A(V)u$ yields the vector which

is closest to $u$ in the $A$-norm and $u \in \mathcal{S}$. Thus the infimum in (5.2) is obtained for $v = -\pi_A(V)u$ and therefore

$$(1-\gamma)\|u\|_A^2 \leq \|u - \pi_A(V)u\|_A^2 \quad \forall u \in \mathcal{S}^\perp \,. \tag{5.3}$$

This yields the desired estimate for $\xi$ since

$$\xi = \min_{x \in \mathcal{S}^\perp \setminus \{0\}} \frac{\|x - \pi_A(V)x\|_A^2}{\|x\|_A^2} \geq (1-\gamma) \,. \tag{5.4}$$

Thus we have proven our main theorem.

THEOREM 5.1. *Let $V \in \mathbb{K}^{n\times m}$ be a matrix such that $V$ has full rank and the weak approximation property (4.1) is fulfilled. Let $\mathcal{S} \subseteq \mathbb{K}^n$ be the subspace spanned by the columns of $V$. Furthermore let $\gamma \in [0,1)$ be the smallest constant such that*

$$|\langle u, v\rangle_A| \leq \gamma \langle u, u\rangle_A^{\frac{1}{2}} \langle v, v\rangle_A^{\frac{1}{2}} \quad \forall u \in \mathcal{S}^\perp,\ \forall v \in \mathcal{S} \,.$$

*Then we can estimate the condition $\kappa = \frac{\mu_1}{\mu_\ell}$ of the matrix $A(I - \pi_A(V))$ by*

$$\kappa \leq \frac{K}{(1-\gamma)} \,. \tag{5.5}$$

*Proof.* Follows from (4.2) and (5.4). □

**6. Deflation Subspaces for $M$-Matrices.** The classical algebraic multigrid method [10, 13] is constructed for the case that $A \in \mathbb{R}^{n\times n}$ is a $M$-matrix, i.e. $A$ is symmetric positive definite and $a_{ij} \leq 0$ for $i \neq j$. In this section we show that we can use the prolongation operators constructed in the classical algebraic multigrid method as the operator $V$ which spans the deflation subspace $\mathcal{S}$.

The construction of those operators is done by inspecting the graph $G(A)$ of a matrix $A \in \mathbb{K}^{n\times n}$ which is given by

$$\begin{aligned} G(A) = (W, E) \quad \text{where} \quad & W = \{1, 2, \ldots, n\} \\ & E = \{(i,j) \in W \times W : a_{ij} \neq 0, i \neq j\} \,. \end{aligned}$$

The neighborhood of a node $i \in W$ is given by

$$N_i := \{j \in W : (i,j) \in E\} \,.$$

To construct the prolongation operator, or the deflation subspaces respectively, we split the variables $W$ in *coarse* and *fine* variables $\mathcal{C}$ and $\mathcal{F}$, such that $W = \mathcal{C} \dot\cup \mathcal{F}$ and $N_i \cap \mathcal{C} \neq \emptyset$ for $i \in \mathcal{F}$. The coarse variables have a direct representation on the coarse grid, or the deflation subspace respectively, while the fine variables interpolate from the coarse ones. For simplicity of notation assume that the variables in $\mathcal{C}$ have a smaller index than those in $\mathcal{F}$, i.e. $\mathcal{C} = \{1, 2, \ldots, m\}$, $\mathcal{F} = \{m+1, m+2, \ldots, n\}$. For every fine variable $i \in \mathcal{F}$ choose a set of variables $P_i \subseteq N_i \cap \mathcal{C}$. The value for the variable $i$ is then interpolated from the variables in $P_i$. Defining the interpolation weights

$$w_{ik} = \alpha_i \frac{-a_{ik}}{a_{ii}} \quad \text{with} \quad \alpha_i = \frac{\sum_{k\in N_i} a_{ik}}{\sum_{k \in P_i} a_{ik}} \quad \text{for} \quad i \in \mathcal{F} \tag{6.1}$$

yields the prolongation (or interpolation) operator $V \in \mathbb{R}^{n\times m}$ by

$$\left(Ve^c\right)_i = \begin{cases} e_i^c & \text{for } i \in \mathcal{C} \\ \sum_{k\in P_i} w_{ik} e_k^c & \text{for } i \in \mathcal{F} \end{cases}. \tag{6.2}$$

Under the assumption that the $\mathcal{C}/\mathcal{F}$-splitting is reasonably well chosen the classical multigrid theory yields an estimate for the constant $K$ of the weak approximation property (4.1).

THEOREM 6.1. *Let $A \in \mathbb{R}^{n\times n}$ be a symmetric weakly diagonally dominant $M$-matrix, i.e. $a_{ij} \le 0$ for $i \ne j$ and $\sum_j a_{ij} \ge 0$ for all $i$. If for fixed $\tau \ge 1$ a $\mathcal{C}/\mathcal{F}$-splitting exists such that for each $i \in \mathcal{F}$ there is a $P_i \in \mathcal{C} \cap N_i$ with*

$$\sum_{k\in P_i} |a_{ik}| \ge \frac{1}{\tau} \sum_{j\in N_i} |a_{ij}| \quad \text{for } i \in \mathcal{F} \tag{6.3}$$

*then there exists a function $R : \mathbb{R}^n \to \mathbb{R}^m$ such that the operator $V$ from (6.2) fulfills*

$$\|e - VR(e)\|_D^2 \le \tau \langle Ae, e\rangle$$

*where $D = \operatorname{diag} A$ is the diagonal matrix containing the diagonal values $a_{ii}$ of $A$.*

*Proof.* Define the function $R : \mathbb{R}^n \to \mathbb{R}^m$ as

$$e_i^c = \left(R(e)\right)_i = e_i \quad \text{for} \quad i \in \mathcal{C} = \{1, \dots, m\}.$$

then the result follows from [13, Theorem A.4.3]. ∎

COROLLARY 6.2. *Under the same assumptions as in theorem 6.1 we have that $V$ fulfills the weak approximation property (4.1) with*

$$K = \frac{\|A\|}{\min_i a_{ii}} \tau.$$

*Proof.* Directly follows from the fact that $(\min_i a_{ii}) \|x\|_2^2 \le \|x\|_D^2$ for $x \in \mathbb{R}^n$ and theorem 6.1. ∎

EXAMPLE 6.3. *Let $N \in \mathbb{N}$ be odd. Consider the block tridiagonal matrix*

$$A = \begin{bmatrix} B & C & & \\ C & B & \ddots & \\ & \ddots & \ddots & C \\ & & C & B \end{bmatrix} \in \mathbb{R}^{N^2\times N^2} \tag{6.4}$$

*with $B, C \in \mathbb{R}^{N\times N}$,*

$$B = \begin{bmatrix} 8 & -1 & & \\ -1 & 8 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 8 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} -1 & -1 & & \\ -1 & -1 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & -1 \end{bmatrix}$$

*The graph $G(A)$ of $A$ is a regular $N \times N$ grid with added diagonal connections, see figure 6.1. We set $\mathcal{C}$ as the variables in odd rows and columns ($\square$), $\mathcal{F}$ as the remaining*

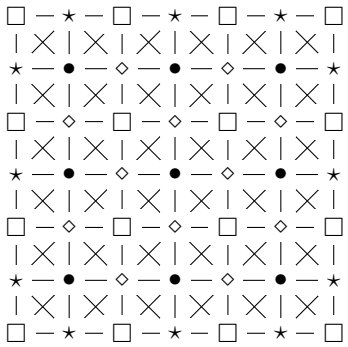

FIGURE 6.1. *The graph of $A$ split into $\mathcal{C}$ and $\mathcal{F}$.*

| | $A$ | $A(I-\pi_A(V))$ | |
|---|---|---|---|
| $\lambda_1$ | 0.0577 | 6.0708 | $\mu_1$ |
| $\lambda_n$ | 11.9616 | 11.9586 | $\mu_\ell$ |
| $\kappa$ | 207.3403 | 1.9698 | $\kappa$ |
| $K$ | 1.4915 | | $K$ |

TABLE 7.1
*Condition of the linear systems for $N = 2^5 - 1$.*

*variables ($\diamond$, $\bullet$ and $\star$).*

*The requirement* (6.3) *is equivalent to $\tau \geq \alpha_i$ for $i \in \mathcal{F}$. A straight forward computation yields $\alpha_i = 2$ for $i \in \bullet$, $\alpha_i = 4$ for $i \in \diamond$ and $\alpha_i = \frac{5}{2}$ for $i \in \star$. Hence $\tau = 4$ fulfills* (6.3)*. Due to Gershgorin's circle theorem the eigenvalues $\lambda_i$ of $A$ fulfill $\lambda_i \in [0, 16]$ and thus $\|A\| \leq 16$ and due to* (6.2) *the weak approximation property is fulfilled for $K = 8$.*

**7. Numerical Experiments.** We consider the matrix $V$ from (6.2) with $A$, $\mathcal{C}$ and $\mathcal{F}$ from example 6.3. If $x$, with $\|x\| = 1$, is an eigenvector corresponding to the eigenvalue $\lambda$, equation (4.1) simplifies to $\|x - VR(x)\|_2^2 \leq (K/\|A\|)\lambda$. Hence

$$K \geq \tfrac{\|A\|}{\lambda}\|x - VR(x)\|_2^2 \,. \tag{7.1}$$

We choose $R(x) := x_c$ such that $\|x - Vx_c\|_2^2$ is minimized and compute the right hand side of (7.1) for all eigenvectors of the matrix $A$. The maximum value over all of those right hand sides is our estimate for $K$. Furthermore we compute the largest and smallest eigenvalues and the condition number of $A$ and $A(I - \pi_A(V))$ numerically. The results for $N = 2^5 - 1$ are given in table 7.1. Equation (5.5) is equivalent to $\gamma \geq (1 - \frac{K}{\kappa})$. Thus we can estimate by the values from table 7.1 that $\gamma \geq 0.2428$.

In example 6.3 we have seen that the constant $K$ is independent of the grid size $N$. Thus we expect the method to converge in a constant number of iterations under the assumption that the abstract angle $\gamma$ is independent of $N$. We choose a random right hand side $b$ such that the solution $x$ fulfills $\|x\| = 1$. Then we run the deflated cg method [12] until the residual $r_i$ of the $i^{\text{th}}$ iterate satisfies $\|r_i\| \leq 10^{-6}$. The number of iterations is listed in table 7.2 where we observe the expected behavior.

**8. Conclusions.** In this paper we derived a convergence estimate for deflation methods based on algebraic multigrid theory. Furthermore we have shown that it is

| $p$ | Iterations | Residual | Error |
|---|---|---|---|
| 4 | 8 | $5.64429 \cdot 10^7$ | $1.53486 \cdot 10^7$ |
| 5 | 8 | $8.4304 \cdot 10^7$ | $4.64197 \cdot 10^7$ |
| 6 | 9 | $5.27683 \cdot 10^7$ | $1.63928 \cdot 10^6$ |
| 7 | 9 | $6.11646 \cdot 10^7$ | $5.74174 \cdot 10^6$ |
| 8 | 9 | $6.36346 \cdot 10^7$ | $2.56345 \cdot 10^5$ |
| 9 | 9 | $6.57814 \cdot 10^7$ | $6.60374 \cdot 10^5$ |

TABLE 7.2
*Number of iterations where $N = 2^p - 1$.*

possible to construct deflation based methods such that grid size independent convergence is obtained. Thus, this major advantage of multigrid methods can be transferred to deflation methods. Since deflation methods do not need the construction of an appropriate smoother, they can be more advantageous than multigrid methods in the case when there is no good smoother readily available.

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