COMBINATION PRECONDITIONING OF SADDLE POINT SYSTEMS FOR POSITIVE DEFINITENESS

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Abstract. Preconditioned iterative methods in nonstandard inner products for saddle point systems have recently received attention. Krzyżanowski (*Numer. Linear Algebra Appl.* 2011; **18**:123–140) identified a two-parameter family of preconditioners in this context and Stoll and Wathen (*SIAM J. Matrix Anal. Appl.* 2008; **30**:582–608) proposed combination preconditioning, where two different preconditioners—for each of which the preconditioned saddle point matrix is self-adjoint with respect to an inner product—can be blended to create additional preconditioners and associated bilinear forms or inner products. If a preconditioned saddle point matrix is nonsymmetric but self-adjoint with respect to a nonstandard inner product a MINRES-type method ($\mathcal{W}$-PMINRES) can be applied in the relevant inner product. If the preconditioned matrix is also positive definite with respect to this inner product a more efficient CG-like method ($\mathcal{W}$-PCG) can be used reliably. We provide explicit expressions for the combination of certain Krzyżanowski preconditioners and prove the rather counterintuitive result that the combination of two specific preconditioners for which only $\mathcal{W}$-PMINRES can be reliably used leads to a preconditioner for which, for certain parameter choices, $\mathcal{W}$-PCG is applicable. That is, the resulting preconditioned saddle point matrix is positive definite with respect to an inner product. This combination preconditioner outperforms either of the two preconditioners from which it is formed for a number of test problems.

1. Introduction.

Consider the real symmetric saddle point system

$$\mathcal{A}x = \begin{bmatrix} A & B^T \\ B & -C \end{bmatrix} x = b, \tag{1.1}$$

where $A \in \mathbb{R}^{n\times n}$ is symmetric positive definite, $C \in \mathbb{R}^{m\times m}$ is symmetric positive semidefinite, B has full rank and $m \leq n$. Under these assumptions $\mathcal{A}$ is invertible [2, Theorem 3.1]. Saddle point systems of the form (1.1) arise in a vast number of applications including constrained optimization, computational fluid dynamics and mixed finite element discretizations of elliptic PDEs [2, Section 2]. Often $\mathcal{A}$ is large and sparse and it is natural in these instances to solve the saddle point system by a preconditioned Krylov subspace method.

Many preconditioners $\mathcal{P} \in \mathbb{R}^{(n+m)\times(n+m)}$ for saddle point problems have been proposed, surveys of which can be found in, for example, [2, 4]. If $\mathcal{P}$ is symmetric positive definite the resulting system can be solved by preconditioned MINRES. However, many effective preconditioners are nonsymmetric or symmetric indefinite. Although it is possible to apply a nonsymmetric Krylov subspace method to the preconditioned system, it may be appealing to turn instead to a Krylov method in a nonstandard inner product on $\mathbb{R}^{n+m}$,

$$\langle x, y\rangle_{\mathcal{W}} = y^T \mathcal{W} x, \tag{1.2}$$

where $x, y \in \mathbb{R}^{n+m}$ and $\mathcal{W} \in \mathbb{R}^{(n+m)\times(n+m)}$ is symmetric positive definite.

For certain preconditioners, $\mathcal{P}^{-1}\mathcal{A}$ is self-adjoint with respect to a known inner product $\langle\cdot,\cdot\rangle_{\mathcal{W}}$, i.e.,

$$\langle \mathcal{P}^{-1}\mathcal{A}x, y\rangle_{\mathcal{W}} = \langle x, \mathcal{P}^{-1}\mathcal{A}y\rangle_{\mathcal{W}} \text{ for all } x, y \in \mathbb{R}^{n+m},$$

or, equivalently,

$$\mathcal{W}\mathcal{P}^{-1}\mathcal{A} = \mathcal{A}^T\mathcal{P}^{-T}\mathcal{W}, \tag{1.3}$$

so that $\mathcal{W}\mathcal{P}^{-1}\mathcal{A}$ is symmetric. The positive definiteness of $\mathcal{W}$ means that $\mathcal{P}^{-1}\mathcal{A}$ is similar to a real symmetric matrix and so has real eigenvalues [21]. Moreover, MINRES in $\langle \cdot, \cdot \rangle_{\mathcal{W}}$ ($\mathcal{W}$-PMINRES) can be applied to the preconditioned system (see [20] for details). If, additionally,

$$\langle \mathcal{P}^{-1}\mathcal{A}x, x \rangle_{\mathcal{W}} > 0 \tag{1.4}$$

for all nonzero $x \in \mathbb{R}^{n+m}$ we say that $\mathcal{P}^{-1}\mathcal{A}$ is positive definite with respect to $\langle \cdot, \cdot \rangle_{\mathcal{W}}$. It is straightforward to show that this condition is equivalent to positive definiteness of $\mathcal{W}\mathcal{P}^{-1}\mathcal{A}$ with respect to the Euclidean inner product. Moreover, if $\mathcal{P}^{-1}\mathcal{A}$ is self-adjoint and positive definite with respect to an inner product its eigenvalues must be real and positive [9, Chapter IX]. A conjugate gradient method in $\langle \cdot, \cdot \rangle_{\mathcal{W}}$, $\mathcal{W}$-PCG, can be applied in this situation (see, for example, [1, 20]). The relative residuals of $\mathcal{W}$-PMINRES and the relative error vectors of $\mathcal{W}$-PCG are bounded by polynomials evaluated at the spectrum of $\mathcal{P}^{-1}\mathcal{A}$ (see, for example, [3, 15]). Although these bounds are not necessarily tight, a sufficient condition for fast convergence is that $\mathcal{P}^{-1}\mathcal{A}$ has nicely distributed eigenvalues.

Thus, when $\mathcal{P}^{-1}\mathcal{A}$ is self-adjoint (and potentially positive definite) with respect to the inner product, we can apply a nonstandard Krylov subspace method that minimizes the residual or error with respect to a norm and requires only three-term recurrences. One potential disadvantage is that computations with $\langle \cdot, \cdot \rangle_{\mathcal{W}}$ must be performed at each iteration. Often, however, the inner product depends on the same (typically sparse) blocks as the preconditioner and computational savings can be made by careful consideration of the operations involved [13]. We note that an alternative Krylov subspace method when the preconditioned saddle point matrix is self-adjoint with respect to a symmetric bilinear form (that is not positive definite) is SQMR [8].

Recently, Stoll and Wathen [20] showed that two preconditioners for which the preconditioned coefficient matrix is self-adjoint could be combined to give a new preconditioner for which the preconditioned saddle point matrix is self-adjoint with respect to a symmetric bilinear form that can, in many cases, be made an inner product. One might expect that if one or both of the original preconditioned linear systems is positive definite with respect to an inner product then the combination preconditioner could either inherit the positive definiteness property or lose it. What is perhaps more surprising is that a combination preconditioner can be constructed from two preconditioners for each of which $\mathcal{P}^{-1}\mathcal{A}$ is *indefinite* with respect to an inner product such that the combination preconditioned saddle point matrix is *positive definite* with respect to an inner product. This combination preconditioner is described and investigated here. Although we focus on left preconditioning in this manuscript, it is also possible to develop a combination from two right preconditioners when each preconditioned saddle point matrix is self-adjoint with respect to a symmetric bilinear form.

The remainder of the paper is organized as follows. In Section 2 we examine certain preconditioners for which the preconditioned saddle point matrix is self-adjoint with respect to an inner product. We focus in Section 3 on a combination of two of these preconditioners, for each of which, separately, the preconditioned saddle point matrix is indefinite with respect to an inner product, but for which the combination preconditioned saddle point matrix can be made positive definite. Numerical experiments with this second combination preconditioner are presented in Section 4 while conclusions are made in Section 5. We note that this paper is a shorter version of [16], where further details can be found.

Preconditioner	$\tilde{S}$	c	d	ϵ
Block diagonal (BD)	S_0	0	0	1
Bramble-Pasciak (BP)	$-I$	1	0	-1
Bramble-Pasciak$^+$ (BP$^+$)	S_0	-1	0	1
Schöberl-Zulehner (SZ)	$-S_0$	1	1	1
Schöberl-Zulehner$^+$ (SZ$^+$)	S_0	-1	-1	1

TABLE 2.1

Krzyżanowski parameters for saddle point preconditioners. In all cases the approximation to A in (2.1) is a symmetric positive definite matrix A_0 and S_0 is assumed to be symmetric positive definite.

2. Preconditioners for saddle point matrices. There exist several preconditioners for which the preconditioned linear system can be made self-adjoint with respect to an inner product (see, for example, [5, 6, 14]). Most are instances of the Krzyżanowski preconditioner [13]

$$\mathcal{P} = \begin{bmatrix} I & 0 \\ cBA_0^{-1} & I \end{bmatrix} \begin{bmatrix} A_0 & 0 \\ 0 & \tilde{S} \end{bmatrix} \begin{bmatrix} I & dA_0^{-1}B^T \\ 0 & I \end{bmatrix}, \tag{2.1}$$

where $A_0 \in \mathbb{R}^{n\times n}$ is a symmetric positive definite approximation of A and $\tilde{S} \in \mathbb{R}^{m\times m}$ is a symmetric definite approximation of the (negative) Schur complement $S = BA^{-1}B^T + C$. As shown in [13], if $\mathcal{A}$ is the saddle point matrix in (1.1) the Krzyżanowski preconditioned matrix $\mathcal{P}^{-1}\mathcal{A}$ is self-adjoint with respect to $\langle \cdot, \cdot \rangle_{\mathcal{W}}$, where

$$\mathcal{W} = \epsilon \begin{bmatrix} A_0 - cA & 0 \\ 0 & \tilde{S} + cdBA_0^{-1}B^T + dC \end{bmatrix} \tag{2.2}$$

and $\epsilon = \pm 1$. Each of the preconditioners discussed below is a Krzyżanowski preconditioner, with the choices of $\tilde{S}$, c and d for each given in Table 2.1

One of the best known Krylov subspace methods in a nonstandard inner product for (1.1) is the Bramble-Pasciak (BP) preconditioned conjugate gradient method [5], a block triangular preconditioner that was later modified to include a Schur complement approximation [11]. An alternative is the Schöberl-Zulehner (SZ) constraint preconditioner in [17, 22]. The major drawback of both the BP and SZ preconditioners is that often A_0 must be scaled for $\mathcal{W}$ to define an inner product. This typically requires an approximation to the smallest eigenvalue of $A_0^{-1}A$, the computation of which can be costly. However, when $\mathcal{W}$ defines an inner product, $\mathcal{P}^{-1}\mathcal{A}$ is self-adjoint with respect to this inner product and $\mathcal{W}$-PCG can be applied to the BP- or SZ-preconditioned saddle point matrix [5, 17].

For other preconditioners positive definiteness of A_0 and S_0—the approximations of A and the Schur complement—are sufficient to ensure that $\mathcal{W}$ in (2.2) is positive definite. Perhaps the simplest is the block diagonal (BD) preconditioner [10, 12, 19]

$$\mathcal{P} = \begin{bmatrix} A_0 & 0 \\ 0 & S_0 \end{bmatrix}. \tag{2.3}$$

Indeed, the self-adjointness requirement (1.3) is trivially satisfied since $\mathcal{W}\mathcal{P}^{-1}\mathcal{A} = \mathcal{A}$, the original symmetric saddle point matrix. The indefiniteness of $\mathcal{A}$, however, means that $\mathcal{P}^{-1}\mathcal{A}$ is indefinite with respect to $\langle \cdot, \cdot \rangle_{\mathcal{W}}$ (see (1.4)).

A second preconditioner that does not require scaling in order that $\mathcal{W}$ in (2.2) defines an inner product is the Bramble-Pasciak$^+$ (BP$^+$) preconditioner [20],

$$\mathcal{P} = \begin{bmatrix} A_0 & 0 \\ -B & S_0 \end{bmatrix} \tag{2.4}$$

for which the preconditioned saddle point matrix is again self-adjoint but indefinite with respect to $\langle \cdot, \cdot \rangle_{\mathcal{W}}$. The Schöberl-Zulehner preconditioner can be similarly modified [13, Table 3]

$$\mathcal{P} = \begin{bmatrix} I & 0 \\ -BA_0^{-1} & I \end{bmatrix} \begin{bmatrix} A_0 & 0 \\ 0 & S_0 \end{bmatrix} \begin{bmatrix} I & -A_0^{-1}B^T \\ 0 & I \end{bmatrix}. \tag{2.5}$$

We call this the Schöberl-Zulehner$^+$ (SZ$^+$) preconditioner. However, although $\langle \cdot, \cdot \rangle_{\mathcal{W}}$ always defines an inner product, the following theorem shows that the preconditioned saddle point matrix is never positive definite with respect to an inner product.

THEOREM 2.1. *Let $\mathcal{A}$ in* (1.1)*, with $C = 0$, be left preconditioned by the SZ$^+$ preconditioner* (2.5)*. Then $\mathcal{P}^{-1}\mathcal{A}$ is indefinite with respect to $\langle \cdot, \cdot \rangle_{\mathcal{W}}$, where $\mathcal{W}$ is defined by*

$$\mathcal{W} = \begin{bmatrix} A + A_0 & \\ & BA_0^{-1}B^T + S_0 \end{bmatrix}. \tag{2.6}$$

Since the BD-, BP$^+$- and SZ$^+$-preconditioned saddle point matrices are all self-adjoint but indefinite with respect to inner products we can always use $\mathcal{W}$-PMINRES to solve these preconditioned systems but cannot reliably apply $\mathcal{W}$-PCG. In the next section we see that the BD and BP$^+$ preconditioners can be combined to give a preconditioner for which, for certain parameter choices, $\mathcal{W}$-PCG can be applied.

3. Combination preconditioning. Combination preconditioning [20] allows two preconditioners, for each of which the preconditioned coefficient matrix is self-adjoint with respect to a symmetric bilinear form, to be blended. The result is a new preconditioner and a symmetric bilinear form with respect to which the combination preconditioned coefficient matrix is self-adjoint. The process is controlled by two parameters, for certain choices of which the combination preconditioner is more effective than either of the original preconditioners and is no more costly to apply than the more expensive of the two. The process of combination preconditioning is described in Lemma 3.5 in [20] which we reproduce below using our notation.

LEMMA 3.1. *If $\mathcal{P}_1$ and $\mathcal{P}_2$ are left preconditioners for the symmetric matrix $\mathcal{A}$ for which symmetric matrices $\mathcal{W}_1$ and $\mathcal{W}_2$ exist with $\mathcal{P}_1^{-1}\mathcal{A}$ self-adjoint in $\langle \cdot, \cdot \rangle_{\mathcal{W}_1}$ and $\mathcal{P}_2^{-1}\mathcal{A}$ self-adjoint in $\langle \cdot, \cdot \rangle_{\mathcal{W}_2}$ and if*

$$\alpha \mathcal{P}_1^{-T}\mathcal{W}_1 + \beta \mathcal{P}_2^{-T}\mathcal{W}_2 = \mathcal{P}_3^{-T}\mathcal{W}_3$$

for some matrix $\mathcal{P}_3$ and some symmetric matrix $\mathcal{W}_3$, then $\mathcal{P}_3^{-1}\mathcal{A}$ is self-adjoint in $\langle \cdot, \cdot \rangle_{\mathcal{W}_3}$.

To combine preconditioners described in the previous section, we first apply Lemma 3.1 to two different Krzyżanowski preconditioners, for each of which $\tilde{S} = S_0$. Thus,

$$\mathcal{P}_1 = \begin{bmatrix} I & 0 \\ c_1BA_0^{-1} & I \end{bmatrix} \begin{bmatrix} A_0 & 0 \\ 0 & S_0 \end{bmatrix} \begin{bmatrix} I & d_1A_0^{-1}B^T \\ 0 & I \end{bmatrix} \tag{3.1}$$

and

$$\mathcal{P}_2 = \begin{bmatrix} I & 0 \\ c_2 B A_0^{-1} & I \end{bmatrix} \begin{bmatrix} A_0 & 0 \\ 0 & S_0 \end{bmatrix} \begin{bmatrix} I & d_2 A_0^{-1} B^T \\ 0 & I \end{bmatrix} \tag{3.2}$$

with bilinear forms defined by

$$\mathcal{W}_1 = \epsilon_1 \begin{bmatrix} A_0 - c_1 A & 0 \\ 0 & S_0 + c_1 d_1 B A_0^{-1} B^T + d_1 C \end{bmatrix} \tag{3.3}$$

and

$$\mathcal{W}_2 = \epsilon_2 \begin{bmatrix} A_0 - c_2 A & 0 \\ 0 & S_0 + c_2 d_2 B A_0^{-1} B^T + d_2 C \end{bmatrix}. \tag{3.4}$$

The following theorem shows that under certain conditions a combination preconditioner can be constructed from (3.1) and (3.2) that retains the structure of a Krzyżanowski preconditioner.

THEOREM 3.2. *Let $\mathcal{P}_1$, $\mathcal{W}_1$, $\mathcal{P}_2$ and $\mathcal{W}_2$ be defined by* (3.1), (3.3), (3.2) *and* (3.4), *respectively, where $A, A_0 \in \mathbb{R}^{n\times n}$ and $S_0 \in \mathbb{R}^{m\times m}$ are symmetric positive definite, $C \in \mathbb{R}^{m\times m}$ is symmetric positive semidefinite, $B \in \mathbb{R}^{m\times n}$ has full rank and $n \geq m$. Then, if $c_1 = c_2 = c$,*

$$\mathcal{P} = \begin{bmatrix} I & 0 \\ cBA_0^{-1} & I \end{bmatrix} \begin{bmatrix} \frac{1}{\alpha\delta_1+\beta\delta_2} A_0 & 0 \\ 0 & S_0 \end{bmatrix} \begin{bmatrix} I & (\alpha\delta_1 d_1 + \beta\delta_2 d_2) A_0^{-1} B^T \\ 0 & I \end{bmatrix}$$

is a combination preconditioner formed from $\mathcal{P}_1$ and $\mathcal{P}_2$ for which $\mathcal{P}^{-1}\mathcal{A}$ is self-adjoint with respect to the symmetric bilinear form defined by

$$\mathcal{W} = \begin{bmatrix} (A_0 - cA) & 0 \\ 0 & (\alpha\delta_1 + \beta\delta_2) S_0 + (\alpha\delta_1 d_1 + \beta\delta_2 d_2)(cBA_0^{-1}B^T + C) \end{bmatrix}.$$

Alternatively, if $d_1 = d_2 = 0$, the combination preconditioner

$$\mathcal{P} = \begin{bmatrix} I & 0 \\ (\alpha\delta_1 c_1 + \beta\delta_2 c_2) B A_0^{-1} & I \end{bmatrix} \begin{bmatrix} A_0 & 0 \\ 0 & \frac{1}{\alpha\delta_1+\beta\delta_2} S_0 \end{bmatrix},$$

formed from $\mathcal{P}_1$ and $\mathcal{P}_2$, is such that $\mathcal{P}^{-1}\mathcal{A}$ is self-adjoint with respect to the symmetric bilinear form defined by

$$\mathcal{W} = \begin{bmatrix} (\alpha\delta_1 + \beta\delta_2) A_0 - (\alpha\delta_1 c_1 + \beta\delta_2 c_2) A & 0 \\ 0 & S_0 \end{bmatrix}.$$

Stoll and Wathen [20] introduced a BP-BP$^+$ combination preconditioner and a BP-SZ combination preconditioner, the former of which takes $\beta = 1 - \alpha$ and the latter of which differs from the preconditioner that would be obtained by Theorem 3.2. This highlights that combination preconditioners are not uniquely defined, although the above characterization certainly makes it straightforward to construct combinations of preconditioners that satisfy the conditions of the theorem.

We are interested in combining preconditioners for which the preconditioned coefficient matrix is indefinite with respect to an inner product to obtain a combination preconditioned matrix can be made definite with respect to an inner product. The SZ$^+$ and BP preconditioners cannot be combined using Theorem 3.2 and we have not found a practical way of combining these preconditioners. The BP$^+$ and SZ$^+$ preconditioners can be combined but the resulting combination preconditioned saddle point matrix is never positive definite with respect to an inner product, as we might expect (see [15, Theorem 7]). However, the BP$^+$ and BD preconditioners can be combined using Theorem 3.2 so that, for certain parameter values, $\mathcal{P}^{-1}\mathcal{A}$ is positive definite (and self-adjoint) with respect to an inner product. In this case, we can apply $\mathcal{W}$-PCG to the preconditioned system rather than $\mathcal{W}$-MINRES and the eigenvalues of $\mathcal{P}^{-1}\mathcal{A}$ are all real and positive.

3.1. The BP$^+$-BD combination preconditioner. Theorem 3.2 applied to the BP$^+$ preconditioner (2.4) and the BD preconditioner (2.3), with $c_1 = -1$, $c_2 = 0$, $d_1 = d_2 = 0$ and $\epsilon_1 = \epsilon_2 = 1$, gives that

$$\mathcal{P} = \begin{bmatrix} A_0 & 0 \\ -\frac{\alpha}{\alpha+\beta}B & \frac{1}{\alpha+\beta}S_0 \end{bmatrix} \tag{3.5}$$

and

$$\mathcal{W} = \begin{bmatrix} \alpha(A + A_0) + \beta A_0 & 0 \\ 0 & S_0 \end{bmatrix}. \tag{3.6}$$

As we shall see, for this choice of $\mathcal{P}$ and $\mathcal{W}$ there exist α and β for which $\mathcal{P}^{-1}\mathcal{A}$ is positive definite with respect to an inner product.

THEOREM 3.3. *Let* (1.1)*, where $A \in \mathbb{R}^{n\times n}$ is symmetric positive definite, $B \in \mathbb{R}^{m\times n}$ has full rank, $C \in \mathbb{R}^{m\times m}$ is symmetric positive semidefinite and $n \geq m$, be left preconditioned by the BP$^+$-BD combination preconditioner* (3.5)*, where $A_0 \in \mathbb{R}^{n\times n}$ and $S_0 \in \mathbb{R}^{m\times m}$ are symmetric positive definite, so that $\mathcal{P}^{-1}\mathcal{A}$ is self-adjoint with respect to $\langle\cdot,\cdot\rangle_{\mathcal{W}}$, where $\mathcal{W}$ is defined by* (3.6)*.*

When

I. *$\alpha > 0$ and $\alpha + \beta < 0$, $\mathcal{W}$ defines an inner product, with respect to which $\mathcal{P}^{-1}\mathcal{A}$ is positive definite, if and only if*

$$A_0 < -\frac{\alpha}{\alpha+\beta}A;$$

II. *$\alpha > 0$ and $\alpha + \beta > 0$, $\mathcal{W}$ defines an inner product with respect to which $\mathcal{P}^{-1}\mathcal{A}$ is indefinite;*

III. *$\alpha < 0$ and $\alpha + \beta > 0$, $\mathcal{W}$ defines an inner product if and only if*

$$A_0 > -\frac{\alpha}{\alpha+\beta}A$$

but $\mathcal{P}^{-1}\mathcal{A}$ is indefinite with respect to this inner product;

IV. *$\alpha < 0$ and $\alpha + \beta < 0$, $\mathcal{W}$ does not define an inner product.*

The pivotal result of Theorem 3.3 is that when $\alpha+\beta < 0$ with $\alpha > 0$ it *is* possible to obtain from the BP$^+$ and BD preconditioners—for each of which the preconditioned saddle point matrix is indefinite with respect to an inner product—a combination preconditioner for which $\mathcal{P}^{-1}\mathcal{A}$ is self-adjoint and positive definite with respect to a nonstandard inner product. One of the advantages of positive definiteness is that $\mathcal{W}$-PCG may be reliably applied instead of $\mathcal{W}$-PMINRES. Even for $\mathcal{W}$-PMINRES, the eigenvalues of a $\mathcal{W}$-positive definite preconditioned saddle point matrix lie on the positive real line and, if clustered, might lead to faster convergence than can be achieved for an indefinite system. Indeed, we shall see below that for these parameters convergence of the combination preconditioner for both $\mathcal{W}$-PMINRES and $\mathcal{W}$-PCG is rapid.

Regardless of whether $\mathcal{W}$-PMINRES or $\mathcal{W}$-PCG is applicable, convergence for the BP$^+$-BD preconditioned system depends heavily on the eigenvalues of $\mathcal{P}^{-1}\mathcal{A}$. These can be bounded when $C = 0$, as the following theorem shows.

THEOREM 3.4. *Let (1.1), where $A \in \mathbb{R}^{n\times n}$ is symmetric positive definite, $B \in \mathbb{R}^{m\times n}$ has full rank, $C \in \mathbb{R}^{m\times m}$ is symmetric positive semidefinite and $n \geq m$, be left preconditioned by the BP$^+$-BDW combination preconditioner (3.5), where $A_0 \in \mathbb{R}^{n\times n}$ and $S_0 \in \mathbb{R}^{m\times m}$ are symmetric positive definite. Also, let $\mathcal{W}$ in (3.6) be positive definite, so that $\mathcal{P}^{-1}\mathcal{A}$ is self-adjoint with respect to an inner product. We assume that*

$$0 < \delta \leq \psi = \frac{u^T A_0 u}{u^T A u} \leq \Delta \tag{3.7}$$

and

$$0 \leq \omega = \frac{u^T B^T S_0^{-1} B u}{u^T A u} \leq \Phi \tag{3.8}$$

and that $[\, u^T, v^T \,]^T$ is an eigenvector of $\mathcal{P}^{-1}\mathcal{A}$. Then, if $Bu = 0$, the corresponding eigenvalue of $\mathcal{P}^{-1}\mathcal{A}$, λ^+, is positive and satisfies

$$\frac{1}{\Delta} \leq \lambda^+ \leq \frac{1}{\delta}.$$

Otherwise, when

I. $\alpha > 0$ *but $\alpha + \beta < 0$ the remaining eigenvalues λ^+, which are all positive, satisfy*

$$\frac{1}{\Delta} \leq \lambda^+ \leq \frac{(1+\alpha\Phi) + \sqrt{(1+\alpha\Phi)^2 + 4\delta\Phi(\alpha+\beta)}}{2\delta}$$

or

$$0 < \lambda^+ \leq \frac{(1+\alpha\Phi) - \sqrt{(1+\alpha\Phi)^2 + 4\delta\Phi(\alpha+\beta)}}{2\delta}; \tag{3.9}$$

II. $\alpha + \beta > 0$*, the remaining positive eigenvalues λ^+ of $\mathcal{P}^{-1}\mathcal{A}$ satisfy*

$$\frac{1}{\Delta} \leq \lambda^+ \leq \frac{(1+\alpha\Phi) + \sqrt{(1+\alpha\Phi)^2 + 4\delta\Phi(\alpha+\beta)}}{2\delta}$$

while negative eigenvalues λ^- are bounded by

$$\frac{(1+\alpha\Phi) - \sqrt{(1+\alpha\Phi)^2 + 4\delta\Phi(\alpha+\beta)}}{2\delta} \leq \lambda^- < 0. \tag{3.10}$$

Remark Neither (3.9) nor (3.10) bound the eigenvalues of $\mathcal{P}^{-1}\mathcal{A}$ away from the origin, with the difficulty caused by (3.8). However, it may be possible to bound these eigenvalues for certain applications.

If it is possible to obtain bounds of the form (3.7) and (3.8) without too great an expense we can ascertain a priori the eigenvalue distribution of $\mathcal{P}^{-1}\mathcal{A}$. This often determines the convergence of both $\mathcal{W}$-PMINRES and $\mathcal{W}$-PCG. Moreover, it may be possible to use these bounds to choose α and β (and scale A_0 and S_0 if necessary) to obtain good eigenvalues.

Problem	h	BP$^+$	BD	Comb (α, β)	% reduction
Channel flow	2^{-4}	59	57	43 (1.7,-2)	25
	2^{-5}	95	95	86 (0.7,-0.6)	9
Backward step	2^{-4}	88	83	69 (1.4,-1.6)	17
	2^{-5}	147	148	140 (1.2,-1)	5
Regularized cavity	2^{-4}	52	48	40 (1.2,-1.5)	17
	2^{-5}	88	81	73 (1.9,-2)	10
Colliding flow	2^{-4}	46	41	34 (0.8,-1)	17
	2^{-5}	72	71	56 (1.4,-1.5)	21

TABLE 4.1

Iteration counts for the BP$^+$ preconditioner, the BD preconditioner and best possible BP$^+$-BD combination preconditioner for $\mathcal{W}$-PMINRES. Also included is the percentage reduction in the number of iterations required by the combination preconditioner compared with the better performing of the BP$^+$ and BD preconditioners.

4. Numerical examples. The four test problems to which we apply the BP$^+$-BD combination preconditioner are the Stokes problems in [7, Chapter 5], discretized by Taylor-Hood (Q_2-Q_1) finite elements by the Matlab package IFISS [18]. Since Taylor-Hood elements are stable, $C = 0$ in (1.1). The approximation A_0 is a no-fill incomplete Cholesky factorization, computed by the `ichol` command in Matlab while the Schur complement approximation is the pressure mass matrix computed by Ifiss. Since $\mathcal{P}_{comb}^{-1}\mathcal{A}$ can be made positive definite with respect to an inner product, we apply both $\mathcal{W}$-PMINRES and $\mathcal{W}$-PCG to the combination preconditioned system, while $\mathcal{W}$-PMINRES is used to solve separately the BP$^+$ and BD preconditioned systems.

The termination criterion for both methods it that the preconditioned residual is reduced by a factor of 10^{-6} in the Euclidean norm. We run $\mathcal{W}$-PMINRES and $\mathcal{W}$-PCG on the combination preconditioned systems for all (α, β) pairs in $[-2, 2] \times [-2, 2]$, excluding $\alpha = \beta = 0$. From these choices we select, in Tables 4.1 and 4.2, an (α, β) pair that gives the lowest number of iterations and for which $\mathcal{W}$ defines an inner product (with respect to which $\mathcal{P}^{-1}\mathcal{A}$ is positive definite in the case of $\mathcal{W}$-PCG). For most problems this count is obtained by multiple choices of α and β.

It is clear from Tables 4.1 and 4.2 that the combination preconditioner offers superior performance to either the BP$^+$ preconditioner or the BD preconditioner. The relative reduction in the number of iterations required by the combination preconditioner, in comparison to the better performing of the BD and BP$^+$ preconditioners, is 19.8% on average for $\mathcal{W}$-PMINRES and 20.3% on average for $\mathcal{W}$-PCG. We additionally remark that in many cases the optimal choices of α and β for $\mathcal{W}$-PMINRES and $\mathcal{W}$-PCG coincide and the number of iterations for each method is close. This suggests that $\mathcal{W}$-PMINRES performs best when $P^{-1}A$ is $\mathcal{W}_{comb}$-positive definite. It also appears that for these problems the cheaper $\mathcal{W}$-PCG method is preferable to $\mathcal{W}$-PMINRES.

Convergence plots for the backward step and regularized cavity flows with $h = 2^{-4}$ for the values of α and β listed in Table 4.1 are shown in Figure 4.1. We observe that the $\mathcal{W}$-PCG and $\mathcal{W}$-PMINRES curves are very similar and decrease more rapidly than those of BP$^+$ and BD preconditioned $\mathcal{W}$-MINRES.

5. Conclusions. The Krzyżanowski preconditioner provides a useful framework for examining preconditioners that render a symmetric saddle point matrix self-adjoint with respect to an inner product. Expressions for combinations of certain

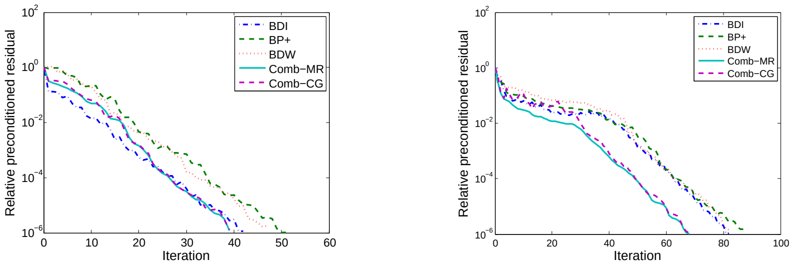
Problem	h	BP$^+$	BD	Comb (α, β)	% reduction
Channel flow	2^{-4}	59	57	43 (1.6,-1.9)	25
	2^{-5*}	94	92	80 (1.9,-2)	13
Backward step	2^{-4}	88	83	70 (1.4,-1.6)	16
	$2^{-5\dagger}$	145	155	118 (1.7,-1.8)	19
Regularized cavity	2^{-4}	52	48	40 (1.2,-1.5)	17
	2^{-5}	88	81	73 (1.4,-1.5)	10
Colliding flow	2^{-4}	46	41	35 (1.2,-1.5)	15
	2^{-5}	72	71	58 (1.5,-1.6)	18

* For positive definiteness of $\mathcal{P}^{-1}\mathcal{A}$ with respect to $\langle\cdot,\cdot\rangle_{\mathcal{W}}$ it was necessary to scale A_0 by 0.9. Thus, $A_0 = 0.9LL^T$, where L is the incomplete Cholesky factor computed by `ichol` in Matlab.

† For positive definiteness of $\mathcal{P}^{-1}\mathcal{A}$ with respect to $\langle\cdot,\cdot\rangle_{\mathcal{W}}$ it was necessary to scale A_0 by 0.7.

TABLE 4.2

Iteration counts for the BP$^+$ preconditioner, the BD preconditioner and best possible BP$^+$-BD combination preconditioner for $\mathcal{W}$-PCG. Also included is the percentage reduction in the number of iterations required by the combination preconditioner compared with the better performing of the BP$^+$ and BD preconditioners.



(a) backward step (b) regularized cavity

FIG. 4.1. *Convergence plots for the BP$^+$-BD combination preconditioner with $\mathcal{W}$-PMINRES and $\mathcal{W}$-PCG for (a) the backward step flow with $\alpha = 1.4$ and $\beta = -1.6$ and (b) for regularized cavity flow with $\alpha = 1.2$ and $\beta = -1.5$.*

Krzyżanowski preconditioners have been derived. From these we constructed the BP$^+$-BD preconditioner. Although, separately, the BP$^+$ and BD preconditioned saddle point matrices were not positive definite with respect to inner products, surprisingly the BP$^+$-BD combination preconditioned saddle point matrix *is* positive definite with respect to an inner product for certain parameter choices. This means that a $\mathcal{W}$-PCG method may be applied to the preconditioned system, iterations of which are cheaper than those of $\mathcal{W}$-PMINRES. More importantly, it highlights the power of combination preconditioning, which constructs a preconditioner $\mathcal{P}$ from two preconditioners, $\mathcal{P}_1$ and $\mathcal{P}_2$, such that $\mathcal{P}^{-1}\mathcal{A}$ can be made positive definite with respect to an inner product when neither $\mathcal{P}_1^{-1}\mathcal{A}$ nor $\mathcal{P}_2^{-1}\mathcal{A}$ are.

The BP$^+$-BD combination preconditioner can, additionally, be more efficient than either the BP$^+$ or BD preconditioners and performs well when the combination preconditioned saddle point matrix is self-adjoint and positive definite with respect to an inner product. It would be interesting to determine other combinations for which positive definiteness can be achieved and to develop efficient ways of selecting good α and β values.

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