Finite element solution of a Schelkunoff vector potential for frequency domain, EM field simulation

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Abstract

A novel method for the 3-D diffusive electromagnetic (EM) forward problem is developed and tested. A Lorentz-gauge, Schelkunoff complex vector potential is used to represent the EM field in the frequency domain and the nodal finite element method is used for numerical simulation. The potential allows for three degrees of freedom per node, instead of four if Coulomb-gauge vector and scalar potentials are used. Unlike the finite-difference method, which minimizes error at discrete points, the finite element method minimizes error over the entire domain cell volumes and may easily adapt to complex topography. Existence and uniqueness of this continuous Schelkunoff potential is proven, boundary conditions are found and a governing equation satisfied by the potential in weak form is obtained. This approach for using a Schelkunoff potential in the finite element method differs from others found in the literature. If the standard weak form of the Helmholtz equation is used, the obtained solution is continuous and has continuous normal derivative across boundaries of regions with different physical properties; however, continuous Schelkunoff potential components do not have continuous normal derivative, divergence of the potential divided by (complex) conductivity and magnetic permeability is continuous instead. Two weak forms of the governing equation are tried. Both of them produce a system matrix that is ill-conditioned and as a result iterative schemes do not converge. A different idea is tried next, instead of representing electric field by a Schelkunoff potential, magnetic field is represented in a similar way. When it is assumed that magnetic permeability μ is constant, a convenient weak form of the governing equation is obtained. This form is tested numerically on a simple model of a conducting prism in a resistive whole space and the code gives a similar results to an independent finite difference code.

1 Introduction

A purpose of this work is to develop a fast and stable method for calculating electromagnetic (EM) fields in a diffusive environment using Finite Element Method (FEM) based on representation of the field by a Schelkunoff potential. The stability of the iterative methods used to solve a linear system resulting from FEM is strongly related to properties of the variational problem used in the formulation. We show that existing variational formulation for Schelkunoff potential [4] does not produce the correct solution to this problem. The present paper develops a new variational formulation, proper in a sense that it leads to a linear system, for which iterative methods are stable.

A motivation for this work is a pursue after a fast algorithm for forward Magnetotelluric (MT) problem. Such problem has to be solved hundreds of times in a 3d inverse MT problem. Nowadays inversion of MT data often need supercomputers, if not, the computations last days or even weeks. The developed theory may find application in every situation when calculation of EM field is needed in a diffusive environment. Although the approach presented here is formulated for Maxwell's equations in frequency domain, the method may be extended to the time domain.

The advantade of using Finite Elements Method (FEM) in comparison with other techniques is that it may be easily adapted to complex boundaries between regions of constant electromagnetic properties, and to topography which is important for magneto-telluric imaging applications. The Schelkunoff potential can provide an appropriate representation of EM field, since the potential is supposed to be continuous [5] and to satisfy Helmholtz equation, with weak form, for which the associated bilinear form is strongly

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coercive. This approach could result in a linear system which is well-posed or not very ill-conditioned and can be efficiently solved with iterative methods. Moreover if the potential is continuous, there is no need for edge elements, simple nodal shape functions can be used.

Unfortunately, to the best knowledge of the authors, there are no results available on existence, uniqueness, and boundary conditions for Schelkunoff potential. Moreover, as we show in the current paper, previous attempts [4] to use Schelkunoff potential for FEM calculation of the EM fields based on weak form of Helmholtz equation, should not produce correct solution.

The present paper develops mathematical foundations of the variational formulation of FEM based on Schelkunoff potential for the purpose of constructing an efficient algorithm for calculation of the EM fields in a diffusive environment or fields propagating in a dissipating medium. In the next section, after giving definition of the electric Schelkunoff potential, we prove the existence and uniqueness theorem using a sequence of lemmas. In the third section two weak forms of the governing equations are suggested. They have been implemented in a FEM algorithm, but unfortunately they produce a system with a very ill-conditioned matrix. Therefore, in the following section, we define the magnetic Schelkunoff potential and introduce a weak form of the governing equation. We show that the bilinear form associated with this equation is strongly coercive, and the matrix is symmetric. In the last section, results of numerical simulations are presented. We show that the iterative scheme used to solve the linear equation converges. The numerical test uses the developed magnetic Schelkunoff potential approach to calculate the field generated by a conductive prism in a resistive whole space. Comparison of the results with calculations done by an independent Finite Difference code, shows very good agreement which provides a verification of the proposed method.

2 Schelkunoff Potential in the literature

Schelkunoff potential, or electric Schelkunoff potential is a vector potential F used to represent the electric field E ([5],[2],[4]) in a form:

$$E = -i\omega F - \nabla\varphi \tag{1}$$

where a relationship between F and φ, called Lorentz Gauge, is imposed:

$$\nabla\left(\frac{\nabla\cdot F}{\hat{\sigma}\mu}\right) = -\nabla\varphi \tag{2}$$

As a result the electric field is represented as:

$$E = -i\omega F + \nabla\left(\frac{\nabla\cdot F}{\hat{\sigma}\mu}\right) \tag{3}$$

Let's consider the electromagnetic field satisfying Maxwell's equations in the frequency domain with the electric source J^{imp}:

$$\left\{\begin{array}{lcl} \nabla\times E &=& -i\omega\mu H \\ \nabla\times H &=& J^{imp} + \hat{\sigma}E\ , \qquad \hat{\sigma} = \sigma + i\omega\epsilon \end{array}\right. \tag{4}$$

Substituting the first equation to the second one, and if (1) is used to represent electric field E, in the region of constant properites $\hat{\sigma}, \mu$ we obtain:

$$\nabla\times(-\nabla\times\frac{1}{i\omega\mu}(-Fi\omega)) = J^{imp} + \hat{\sigma}(-Fi\omega - \nabla\varphi)$$

$$\nabla\times(\nabla\times\frac{1}{\mu}F) = J^{imp} - \hat{\sigma}i\omega F - \hat{\sigma}\nabla\varphi$$

If a well known vector identity (7) is used: the result is:

$$\nabla(\nabla\cdot\frac{1}{\mu}F)) - \nabla\cdot(\nabla(\frac{1}{\mu}F)) = -J^{imp} - \hat{\sigma}i\omega F - \hat{\sigma}\nabla\varphi$$

and if the equation is multiplied by $-\hat{\sigma}\mu$ (it is assumed that $\hat{\sigma}, \mu$ are constant), the Lorentz Gauge (2) is used, the following Helmholtz equation is obtained:

$$\Delta F - i\hat{\sigma}\mu\omega F = \mu J^{imp} \tag{5}$$

Weak form of Helmholtz equation (5) was used in FEM simulations of EM field ([4]) in a situation of piecewise constant $\hat{\sigma}, \mu$ for calculation of electric Schelkunoff potential F.

$$\forall_{j=1,2,3}\forall_{A_j \in H_1(\Omega)} \quad \int_\Omega \nabla A_j \cdot \nabla F_j + i\omega \int_\Omega \hat{\sigma}\mu F_j \cdot A_j = -\int_\Omega \mu J_j^{imp} \cdot A_j \tag{6}$$

Nevertheless this equation imposes conditions on the boundaries between regions of different $\hat{\sigma}, \mu$ listed below:

1. $\forall_{j=1,2,3}$ F_j is continuous
2. $\forall_{j=1,2,3}$ $\frac{\partial}{\partial n} F_j$ is continuous

Investigation of existance and uniqueness of Schelkunoff potential satisfying those conditions is the purpose of this work. As it turns out, with some reasonable assumptions, a continuous Schelkunoff potential (condition 1. is satisfied) exists, yet the condition 2. is not satisfied. As a result there is no Schelkunoff potential that satisfies weak form of Helmholtz equation (6). This result is explained in detail in the next section.

3 Existence and uniqueness of *continuous* electric Schlekunoff potential F

Let us consider an open bounded set $\Omega \subset R^3$, which is divided into a finite number of disjoint open sets $V_j, j \in I$, such that

$$\bigcup_{j \in I} V_j \subset \Omega \subset \bigcup_{j \in I} \overline{V}_j$$

It is assumed that the properties $\hat{\sigma}, \mu$ are constant in each V_j. We may say that the properties $\hat{\sigma}, \mu$ are piecewise constant in Ω.

It is assumed further that all functions considered are $C^\infty(\overline{V_j})$ for $j \in I$, yet the functions may have jumps in value or derivatives across boundaries between sets V_j. It is assumed not only that Maxwell's equations (4) are satisfied in strong sense in each V_j but also that they are satisfied in weak sense in Ω.

Three vector identities will be used. $K, L : R^3 \to R^3, u : R^3 \to R$

$$\nabla \times \nabla \times K = \nabla(\nabla \cdot K) - \nabla \cdot (\nabla K) \tag{7}$$

$$\int_\Omega (\nabla \times K) \cdot L = \int_\Omega K \cdot (\nabla \times L) + \int_{\partial\Omega} (n \times K) \cdot L \tag{8}$$

$$\int_\Omega \nabla u \cdot K = -\int_\Omega u \nabla \cdot K + \int_{\partial\Omega} u(K \cdot n) \tag{9}$$

As a definition of a Schelkunoff potential we will take a vector field that is properly defined in each region V_j, no matter what are the boundary conditions between regions V_j.

Definition 3.1 *An electric Schelkunoff potential is any vector field $F \in C^\infty(\overline{V_j})$ for all $j \in I$ satisfying*

$$E = -i\omega F + \nabla \left(\frac{\nabla \cdot F}{\hat{\sigma}\mu} \right) \quad in\ V_j\ \forall_{j \in I} \tag{10}$$

where E is electric field satisfying Maxwell's Equations (4) in weak sense in Ω.

It turns out that if we assume that the source J^{imp} is piecewise divergence free, then electric field E multiplied by a constant is an electric Schelkunoff potential, which is expressed in the following theorem:

Lemma 3.2 *If $\nabla \cdot J^{imp} = 0$ in V_j for all $j \in I$, then $\nabla \cdot E = 0$ in V_j for all $j \in I$ and*

$$F_E = -\frac{1}{i\omega} E \tag{11}$$

is an electric Schelkunoff potential, as defined in Definition 3.1

Proof As Maxwell's Equations (4) are satisfied in strong sense in each of V_j, then the following equation is satisfied in strong sense:

$$\nabla \times (-\frac{1}{i\omega\mu} \nabla \times E) = J^{imp} + \hat{\sigma} E$$

in each V_j. Taking divergence of both sides, it is obtained:

$$\nabla \cdot J^{imp} = -\nabla \cdot (\hat{\sigma} E) = -\hat{\sigma} \nabla \cdot E$$

and the last equality holds as $\hat{\sigma}$ is constant. As a result if $\nabla \cdot J^{imp} = 0$, then

$$\nabla \cdot E = 0$$

in each of V_j. If we define F_E according to (11), then

$$\nabla \cdot F_E = \nabla \cdot (-\frac{1}{i\omega} E) = -\frac{1}{i\omega} \nabla \cdot (E) = 0$$

so the equation (10), defining electric Schelkunoff potential, is satisfied as

$$-i\omega F_E + \nabla \left(\frac{\nabla \cdot F_E}{\hat{\sigma}\mu} \right) = -i\omega F_E + 0 = -i\omega (-\frac{1}{i\omega} E) = E$$

the equation above is satisfied in strong sense in each of V_j only, but it is enough for F_E to be electric Schelkunoff potential according to Definition 3.1. ∎

The fact that the equation (11) defining electric Schelkunoff potential is linear allows us to state the conditions for the vector field F to be an electric Schelkunoff potential as a condition on the difference $F - F_E$. This is expressed in the following theorem.

Lemma 3.3 *If $\nabla \cdot Jimp = 0$ in each V_j, then F is a Schelkunoff potential if and only if $K = F - F_E = F + \frac{1}{i\omega} E$ satisfies the equation:*

$$-i\omega K + \nabla (\nabla \cdot \frac{K}{\hat{\sigma}\mu}) = 0 \tag{12}$$

in strong sense in each V_j.

And if and only if there exists scalar field φ such that $K = \nabla \varphi$ in strong sense in each V_j and the following equation is satisfied

$$\Delta \varphi - i\omega\hat{\sigma}\mu \, \varphi = 0 \tag{13}$$

in strong sense in each V_j.

Proof As both F and F_E satisfy the equation (10) defining electric Schelkunoff potential, if those equations are substracted the equation (12) is obtained for K.

Moreover if (12) is investigated one can see that in each V_j, K is a gradient of some function, $K = \nabla(-\frac{1}{i\omega\hat{\sigma}\mu} \nabla \cdot K) = \nabla \widetilde{\varphi}$. If the latter is plugged into (12), we obtain:

$$-i\omega \nabla \widetilde{\varphi} + \nabla (\nabla \cdot \frac{\nabla \widetilde{\varphi}}{\hat{\sigma}\mu}) = 0$$

so also

$$\nabla \left(-i\omega \widetilde{\varphi} + (\nabla \cdot \frac{\nabla \widetilde{\varphi}}{\hat{\sigma}\mu}) \right) = 0$$

multiplying by $\hat{\sigma}\mu$ and using the fact that $\hat{\sigma}, \mu$ are constant, the following is obtained:

$$\nabla (\nabla \cdot \nabla \widetilde{\varphi} - i\omega\hat{\sigma}\mu\, \widetilde{\varphi}) = 0$$

which proves that for each $j \in I$ there exists a constant C_j such that

$$\nabla \cdot \nabla \widetilde{\varphi} - i\omega\hat{\sigma}\mu\, \widetilde{\varphi} = C_j$$

in each V_j, so also

$$\nabla \cdot \nabla \widetilde{\varphi} - i\omega\hat{\sigma}\mu \left(\widetilde{\varphi} - \frac{C_j}{i\omega\hat{\sigma}\mu} \right) = 0$$

Let us define $\varphi = \widetilde{\varphi} - \frac{C_j}{i\omega\hat{\sigma}\mu}$. Because in each V_j, $C_j, \hat{\sigma}, \mu$ are constant, we have that in each V_j, $\nabla \widetilde{\varphi} = \nabla \varphi$, so $K = \nabla \varphi$ and (13) is satisfied in strong sense in each V_j. We have proven that if F is a Schelkunoff potential, $K = F - F_E = \nabla \varphi$ for some φ satysfying (13).

Let us now assume that $F = F_E + \nabla \varphi$ and φ satisfies (13). To prove that F is a Schelkunoff potential it is enough to prove that $K = \nabla \varphi$ satisfies (12), which is obtained if gradient of the equation (13) is taken. ∎

Is there a continuous electric Schelkunoff potential?

Let us investigate what boundary conditions on φ it would impose. let us consider two regions V_1, V_2 and a boundary between those regions $\partial V_1 \cap \partial V_2$. Let F_1, F_2 be potentials in V_1, V_2 respectively. Let n be a vector normal to the boundary, pointing towards V_2. Let us split continuity of F into continuity of tangential components and continuity of normal components of F:

$$n \times F_1 = n \times F_2 \tag{14}$$

$$n \cdot F_1 = n \cdot F_2 \tag{15}$$

If F is a Schelkunoff potential, then according to Lemma 3.3, $F = F_E + K = -\frac{1}{i\omega} E + K$. As the tangential component of electric field is continuous, in order for (14) to be satisfied, it is needed that K has continuous tangential components.

$$n \times K_1 = n \times K_2 \iff n \times \nabla \varphi_1 = n \times \nabla \varphi_2$$

The last equation states that the derivative in the tangential direction is the same on both sides of the boundary.

$$\frac{\partial \varphi_1}{\partial t} = \frac{\partial \varphi_2}{\partial t} \tag{16}$$

where $\frac{\partial \varphi_j}{\partial t}$ is the tangential derivative. The derivative in (any) tangential direction. The equation (16) is understood as satisfied for all tangential directions.

We obtain that (14) is equivalent to continuity of the tangential derivative of φ as expressed in the equation (16).

Let us now focus on a condition more difficult to satisfy – continuity of the normal component, the equation (15). Unfortunately the normal component of the electric field is not continuous.

Representing the potential in a form $F = -\frac{1}{i\omega} E + K$ one may see that (15) is equivalent to

$$n \cdot (-\frac{1}{i\omega} E_1 + K_1) = n \cdot (-\frac{1}{i\omega} E_2 + K_2)$$

which is equivalent to

$$n \cdot (K_1 - K_2) = n \cdot (\frac{1}{i\omega} E_1 - \frac{1}{i\omega} E_2)$$

what can be expressed as

$$n \cdot (\nabla \varphi_1 - \nabla \varphi_1) = n \cdot (\frac{1}{i\omega} E_1 - \frac{1}{i\omega} E_2) \tag{17}$$

Or using a different notation:

$$\frac{\partial \varphi_1}{\partial n} - \frac{\partial \varphi_2}{\partial n} = n \cdot (\frac{1}{i\omega} E_1 - \frac{1}{i\omega} E_2) \tag{18}$$

We obtain that the continuity of the normal component of F, (15) is equivalent to a jump in the normal derivative of φ, dictated by values of E_1 and E_2, the jump is given in (18).

Armed with those results we are ready to prove the main theorem of this paper:

Theorem 3.4 *If the following assumptions are satisfied:*

- *Maxwell's Equations (4) are satisfied for E, H in weak sense in an open, bounded region Ω*
- *$\hat{\sigma}, \mu$ are constant in each V_j, $\hat{\sigma}$ is real and*

$$0 < \sigma_{min} \leq \hat{\sigma}_j \leq \sigma_{max}, \ 0 < \mu_{min} \leq \mu_j \leq \mu_{max} \ \textit{ for all } j \in I$$

- *$\nabla \cdot J^{imp} = 0$ inside each V_j*

Then there exists a continuous Schelkunoff potential satysfying

$$E = -i\omega F + \nabla \left(\frac{\nabla \cdot F}{\hat{\sigma}\mu} \right) \tag{19}$$

$$\frac{\nabla \cdot F}{\hat{\sigma}\mu} \ \textit{ is continuous across boundaries between sets } V_j \tag{20}$$

so (19) is satisfied in the sense of distributions. Moreover if

$$\nabla \cdot F|_{\partial\Omega} = 0 \tag{21}$$

then electric Schelkunoff potential F is unique. If

$$\left. \frac{\partial}{\partial n} (\nabla \cdot F) \right|_{\partial\Omega} = 0 \tag{22}$$

then electric Schelkunoff potential F is unique.

Proof Let us consider a Schlkunoff potential F, so the vector field such that $F = F_E + K = F_E + \nabla\varphi$ and (13) is satisfied in strong sense in each V_j. Because $F = F_E + \nabla\varphi = -\frac{1}{i\omega}E + \nabla\varphi$ and $\nabla \cdot E = 0$ in V_j, then we have

$$\nabla \cdot F = \nabla \cdot (-\frac{1}{i\omega}E + \nabla\varphi) = \nabla \cdot \nabla\varphi \tag{23}$$

so using the equation (13)

$$\frac{\nabla \cdot F}{\hat{\sigma}\mu} = \frac{\nabla \cdot \nabla\varphi}{\hat{\sigma}\mu} = i\omega\varphi \tag{24}$$

As a result, φ being continuous is equivalent to $\frac{\nabla \cdot F}{\hat{\sigma}\mu}$ being continuous. and as we want (20) to be satisfied, we will consider φ continuous, so $\varphi \in H^1(\Omega)$.

Let us now find a weak equation for φ that will result in a continuous Schelkunoff potential F. We want to find φ such that (13), (16), (18) is satisfied.

It is useful to notice that (16) will be satisfied if only $\varphi_1 = \varphi_2$ on $\partial V_1 \cap \partial V_2$, so as φ is a continuous function in Ω then (16) is satisfied. Let's multipy (13) by a conjugated test function and integrate by parts:

$$\sum_{j\in I} \int_{V_j} \Delta\varphi\bar{\xi} - i\omega \sum_{j\in I} \int_{V_j} \hat{\sigma}\mu\,\varphi\,\bar{\xi} = 0$$

$$-\sum_{j\in I} \int_{V_j} \nabla\varphi \cdot \nabla\bar{\xi} + \sum_{j\in I} \int_{\partial V_j} (\frac{\partial}{\partial n_j}\varphi_j)\bar{\xi} - i\omega \sum_{j\in I} \int_{V_j} \hat{\sigma}\mu\,\varphi\,\bar{\xi} = 0$$

where n_j denotes a unit outward normal on ∂V_j. Let us now analyze the term with integration on ∂V_j

$$\sum_{j\in I} \int_{\partial V_j} (\frac{\partial}{\partial n_j}\varphi_j)\bar{\xi} \ = \sum_{j_1\in I,\, j_2\in I, j_1\neq j_2} \int_{\partial V_{j_1}\cap\partial V_{j_2}} \left(\frac{\partial}{\partial n_{j_1}}\varphi_{j_1} + \frac{\partial}{\partial n_{j_2}}\varphi_{j_2} \right)\bar{\xi} + \int_{\partial\Omega} (\frac{\partial}{\partial n}\varphi)\bar{\xi}$$

and using (18), the latter equals:

$$= \sum_{j_1\in I,\, j_2\in I, j_1\neq j_2} \int_{\partial V_{j_1}\cap\partial V_{j_2}} \frac{1}{i\omega} \left(n_{j_1} \cdot E_{j_1} + n_{j_2} \cdot E_{j_2} \right) \bar{\xi} + \int_{\partial\Omega} \frac{\partial}{\partial n}\varphi\bar{\xi}$$

Let's define a function

$$f = \frac{1}{i\omega}(n_{j_1} \cdot E_{j_1} + n_{j_2} \cdot E_{j_2})$$

on the set $D = \bigcup_{j_1 \in I,\, j_2 \in I, j_1 \neq j_2} \partial V_{j_1} \cap \partial V_{j_2}$ consisting of boundaries between V_{j_1} and V_{j_2}, the equation for φ is obtained:

$$\int_\Omega \nabla\varphi \cdot \nabla\overline{\xi} + i\omega \int_\Omega \hat{\sigma}\mu\, \varphi\, \overline{\xi} = \int_D f\overline{\xi} + \int_{\partial\Omega} (\frac{\partial}{\partial n}\varphi)\overline{\xi} \tag{25}$$

If we assume $\frac{\partial}{\partial n}\varphi = 0$ on $\partial\Omega$, then the term $\int_{\partial\Omega}(\frac{\partial}{\partial n}\varphi)\overline{\xi}$ disappears.

If we assume $\varphi, \xi \in H_0^1(\Omega)$, so $\xi = 0$ on $\partial\Omega$, the term $\int_{\partial\Omega}(\frac{\partial}{\partial n}\varphi)\overline{\xi}$ disappears.

For both Dirichlet ($\varphi|_{\partial\Omega} = 0$) and Neumann ($\frac{\partial}{\partial n}\varphi|_{\partial\Omega} = 0$) boundary conditions for φ, the weak equation for φ is:

$$\int_\Omega \nabla\varphi \cdot \nabla\overline{\xi} + i\omega \int_\Omega \hat{\sigma}\mu\, \varphi\, \overline{\xi} = \int_D f\overline{\xi} \tag{26}$$

For Dirichlet boundary conditions $\varphi, \xi \in H_0^1(\Omega)$ and for Neumann boundary conditions $\varphi, \xi \in H^1(\Omega)$. Let us analyze the equation (26).

With our simplifying assumption that all the functions are $C^\infty(\overline{V_j})$ for all $j \in I$, function f defined by values of E is continuous on D and as D is bounded, so is f. As a result $f \in L^2(D)$, and if $\xi \in H^1(\Omega)$, then $\xi|_D \in L^2(D)$, so $\xi \to \int_D f\overline{\xi}$ is a bounded linear functional on $H^1(\Omega)$ (and on $H_0^1(\Omega)$).

Let us analyze the left hand side of equation (26). It is a bilinear form $B(\varphi, \xi)$, which is bounded

$$\begin{aligned}
|B(\varphi,\xi)|^2 &= \left| \int_\Omega \nabla\varphi \cdot \nabla\overline{\xi} + i\omega \int_\Omega \hat{\sigma}\mu\, \varphi\, \overline{\xi} \right|^2 \\
&\leq 2\left(\int_\Omega |\nabla\varphi|\,|\nabla\xi|\right)^2 + 2\omega^2\sigma_{max}^2\mu_{max}^2 \left(\int_\Omega |\varphi|\,|\xi|\right)^2 \\
&\leq 2||\Delta\varphi||_0^2\,||\Delta\xi||_0^2 + 2\omega^2\sigma_{max}^2\mu_{max}^2||\varphi||_0^2\,||\xi||_0^2 \\
&\leq (2 + 2\omega^2\sigma_{max}^2\mu_{max}^2)\left(||\Delta\varphi||_0^2\,||\Delta\xi||_0^2 + ||\varphi||_0^2\,||\xi||_0^2\right) \\
&\leq (2 + 2\omega^2\sigma_{max}^2\mu_{max}^2)\left(||\Delta\varphi||_0^2 + ||\varphi||_0^2)(||\Delta\xi||_0^2 + ||\xi||_0^2\right) \\
&= (2 + 2\omega^2\sigma_{max}^2\mu_{max}^2)||\varphi||_1^2||\xi||_1^2
\end{aligned}$$

Moreover this bilinear form is coercive:

$$|B(\xi,\xi)| = \left| \int_\Omega \nabla\xi \cdot \nabla\overline{\xi} + i\omega \int_\Omega \hat{\sigma}\mu\, \xi\, \overline{\xi} \right| = \left| \int_\Omega |\nabla\xi|^2 + i\omega \int_\Omega \hat{\sigma}\mu\, |\xi|^2 \right|$$

And because the first term is purely real and the second term is purely imaginary ($\omega, \hat{\sigma}, \mu \in R$), then

$$\geq \frac{1}{\sqrt{2}}\left(\int_\Omega |\nabla\xi|^2 + \omega \int_\Omega \hat{\sigma}\mu\, |\xi|^2\right) \geq \frac{1}{\sqrt{2}}\left(\int_\Omega |\nabla\xi|^2 + \omega\sigma_{min}\mu_{min} \int_\Omega |\xi|^2\right)$$

$$\geq \frac{1}{\sqrt{2}} min(1, \omega\sigma_{min}\mu_{min})\left(\int_\Omega |\nabla\xi|^2 + \int_\Omega |\xi|^2\right) = \frac{1}{\sqrt{2}} min(1, \omega\sigma_{min}\mu_{min})||\xi||_1^2$$

To sum up, the left hand side of the equation (26) is a bounded coercive bilinear form on $H^1(\Omega) \times H^1(\Omega)$ (and on $H_0^1(\Omega) \times H_0^1(\Omega)$), the right hand side is a bounded linear functional on $H^1(\Omega)$ (and on $H_0^1(\Omega)$), so from Lax-Milgram theorem there exists a (unique) $\varphi \in H^1(\Omega)$ (or $\varphi \in H_0^1(\Omega)$) satisfying equation (26).

It is an easy exercise to prove that if $\varphi \in C^\infty(\overline{V_j})$ for all $j \in I$ and $\varphi \in H^1(\Omega)$ satisfy equation (26) for all $\xi \in H_0^1(\Omega)$ then (13), (16), (18) is satisfied.

Let us now relate boundary conditions (21) (22) with boundary conditions $\varphi|_{\partial\Omega} = 0$ and $\frac{\partial}{\partial n}\varphi|_{\partial\Omega} = 0$ respectively. From (24)

$$\nabla \cdot F|_{\partial\Omega} = 0 \Leftrightarrow \varphi|_{\partial\Omega} = 0 \tag{27}$$

and because $\hat{\sigma}, \mu$ are constant in each V_j we obtain

$$\frac{\partial}{\partial n}\nabla \cdot F|_{\partial\Omega} = 0 \Leftrightarrow \frac{\partial}{\partial n}\varphi|_{\partial\Omega} = 0 \tag{28}$$

To sum up the existence and uniqueness of $\varphi \in H_0^1(\Omega)$ satisfying (25) for all $\xi \in H_0^1(\Omega)$ is equivalent to existence and uniqueness of a continuous electric Schelkunoff potential satisfying (20) and (21).

Existence and uniqueness of $\varphi \in H^1(\Omega)$ satisfying (25) for all $\xi \in H^1(\Omega)$ is equivalent to existence and uniqueness of a continuous Schelkunoff potential satisfying (20) and (22).

4 Proper weak forms of the governing equation for the electric Schelkunoff potential F

To be able to use Finite Element Method for calulation of EM field, a weak form of the governing equation satisfied by the electric Schelkunoff potential is needed. Two such equations, which impose proper conditions on the solution on the boundaries between regions of different $\hat{\sigma}, \mu$, are proposed.

$F \in (H^1(\Omega))^3$, $n \times F|_{\partial\Omega} = (-\frac{1}{i\omega}E)$, $\phi \in H^1(\Omega)$, $\phi|_{\partial\Omega} = 0$

$$\begin{array}{lrcl} \forall_{A \in (H^1(\Omega))^3,\, n\times A|_{\partial\Omega}} & \int_\Omega \frac{1}{\mu}(\nabla \times F)\cdot(\nabla \times A) + i\omega \int_\Omega \hat{\sigma} F \cdot A + \int_\Omega \hat{\sigma}\nabla\phi \cdot A & = & \int_\Omega J^{imp}\cdot A \\ \forall_{a \in H^1(\Omega),\, a|_{\partial\Omega}=0} & \int_\Omega a \cdot \frac{1}{\hat{\sigma}\mu}\nabla \cdot F & = & -\int_\Omega \phi\, a \end{array}$$

And the second equation is

$F \in \bigcup_j C^2(\overline{V_j})$, $n \times F|_{\partial\Omega} = (-\frac{1}{i\omega}E)$

$$\begin{array}{ll} \forall_{A \in (H^1(\Omega))^3,\, n\times A|_{\partial\Omega}} & \int_\Omega \frac{1}{\mu}(\nabla \times F)\cdot(\nabla \times A) + \int_\Omega \frac{1}{\mu}(\nabla \cdot F)(\nabla \cdot A) + i\omega \int_\Omega \hat{\sigma} F\cdot A \\ & -\sum_{j\neq k}\int_{\partial V_j \cap \partial V_k} \frac{1}{2}\left[\frac{\nabla\cdot F_j}{\hat{\sigma}_j\mu_j} + \frac{\nabla \cdot F_k}{\hat{\sigma}_k\mu_k}\right](\hat{\sigma}_j A\cdot n_j + \hat{\sigma}_k A \cdot n_k) = \int_\Omega J^{imp}\cdot A \end{array}$$

Both equations were implemented and unfortunately both of them produce a system matrix that is not symmetric and very ill-conditioned. Preconditioners tried (ILU0, ILUT, ILUTP) are not stable, and although a couple of different algorithms were tried (GMRES, QMR, CGNR, BICG, BICGSTAB), the iterative schemes do not converge to the solution of a linear system.

Because of this difficulty a different approach has been proposed.

5 Magnetic Schelkunoff Potential G

Instead of representing E field by (3) a similar representation for magnetic field H may be used, namely with a magnetic Schelkunoff potential G. This representation is mentioned in [5], yet according to the knowledge of the authors, no numerical simulations were done using this potential.

$$H = G - \nabla\left(\frac{\nabla \cdot G}{\hat{\sigma}\mu i\omega}\right) \tag{29}$$

A similar theorem to Theorem 3.4 of existence and uniqueness of continuous magnetic Schelkunoff potential G for which $\frac{\nabla\cdot G}{\hat{\sigma}\mu}$ is continuous may be forumlated and prove. This time a condition $\nabla \cdot H = 0$ in each V_j (similar to $\nabla \cdot E = 0$ in each V_j) is naturally satisfied due to lack of magnetic monopoles.

The weak form of the governing equation for magnetic potential is much more appealing to use in FEM simulations because the corresponding iterative scheme is stable.

Theorem 5.1 *If $\mu = \mu_0 = const$, then the magnetic Schelkunoff potential G satisfies the equation below:* $G \in (H^1(\Omega))^3$, $G|_{\partial\Omega} = H|_{\partial\Omega}$, $\forall_{A \in (H^1(\Omega))^3,\, n\times A|_{\partial\Omega}}$

$$\int_\Omega \frac{1}{\hat{\sigma}}(\nabla \times G)\cdot(\nabla \times A) + \int_\Omega \frac{1}{\hat{\sigma}}(\nabla \cdot G)(\nabla \cdot A) + i\omega\int_\Omega \mu_0 G\cdot A \quad = \quad \int_\Omega \frac{1}{\hat{\sigma}} J^{imp}\cdot(\nabla \times A)$$

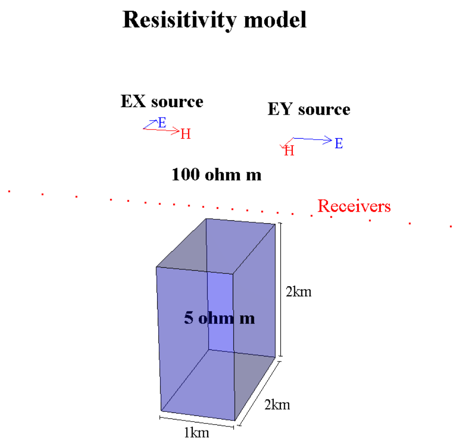
Moreover $\nabla \cdot G = 0$, so in fact the obtained solution G is the magnetic field H. $G = H$.

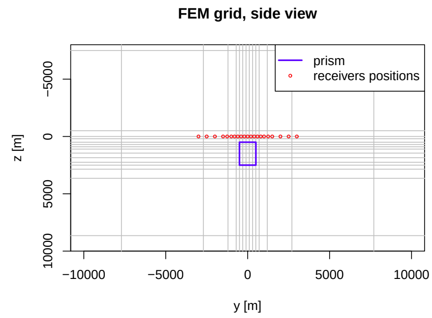
If $\hat{\sigma}$ is real, $0 < \sigma_0 < \hat{\sigma} < \sigma_1 < \infty$, than the bilinear form associated with the equation above is bounded and strongly coercive. The corresponding system matrix is symmetric.

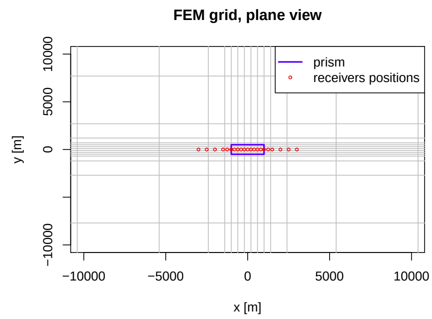
6 Numerical results

The magnetic field calculated for the model described below, is compared with the field calculated by an independent finite difference code.

The model considered is a conductive prism of resistivity $5\Omega \cdot m$ in the whole space of resistivity $100\Omega \cdot m$. The field is calculated 500m above the prism. Just like on the figure below. Second order nodal shape functions for hexahedral elements are used.

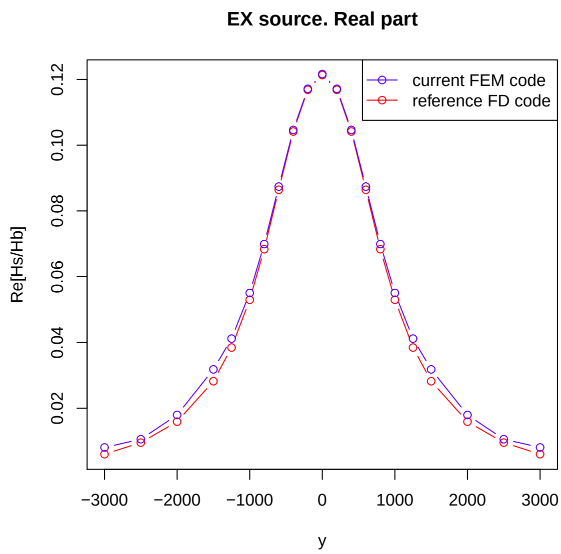


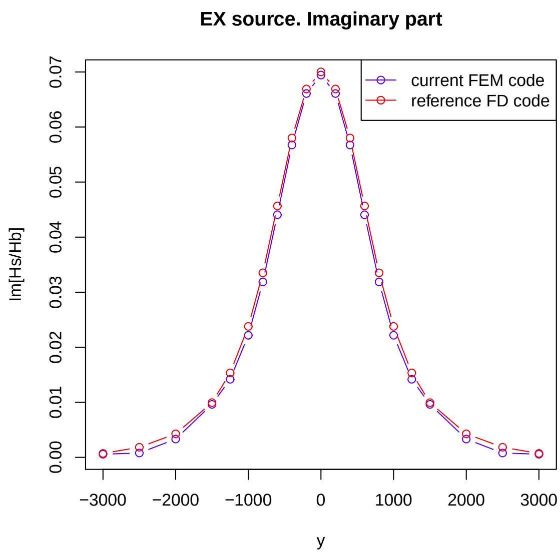


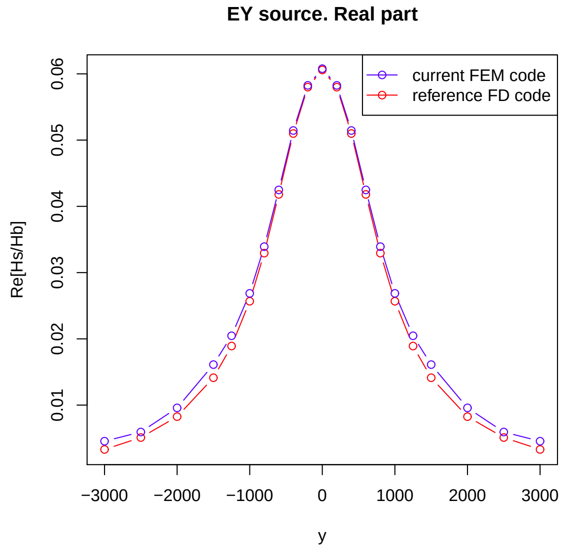


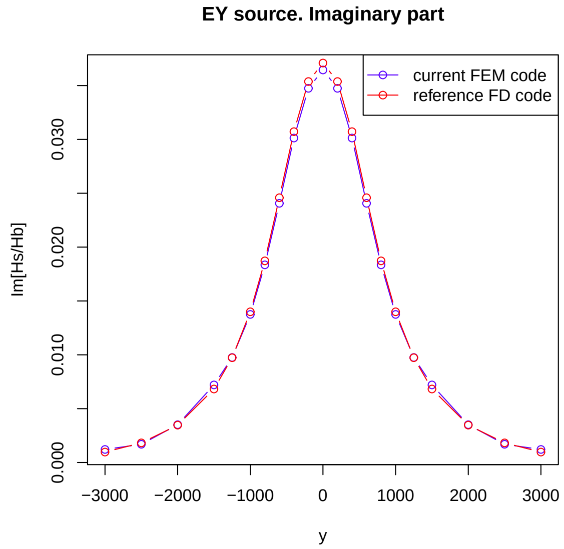
The background field is a plane wave traveling in increasing z direction in the whole space of resistivity $100\Omega \cdot m$ at the frequency $1Hz$. The field calculated by the model is the scattered field due to the presence of the prism. Ratio of the scattered field to the background field is calculated. Two situations are considered:

- plane wave with E field only in X direction
- plane wave with E field only in Y direction









The results of FD code and FEM code solving for G are very similar. The proposed method gives proper values for the magnetic field H. The model tested is very simple, with not big conductivity contrast. It is needed to test the method with bigger contrasts, especially when the boundary air/earth is present. This is a topic for future work.

References

[1] Bochev, P. B., Gunzburger, M. D.(2009) Least-Squares Finite Element Methods, Applied Mathematical Sciences, Vol. 166

[2] Bossavit, A. (1999). On the lorenz gauge. COMPEL - the International Journal for Computation and Mathematics in Electrical and Electronic Engineering, 18(3), 323-336

[3] Bryant, C. F., Emson, C. R. I., & Trowbridge, C. W. (1990). Comparison of lorentz gauge formulations in eddy current computations. IEEE Transactions on Magnetics, 26(2), 430-433.

[4] Um, E. S., Alumbaugh, D. L., & Harris, J. M. (2010). A lorenz-gauged finite-element solution for transient CSEM modeling

[5] Ward, S., Hohmann, G. (1988) Electromagetic Theory for Geophysical Applications. Electromagnetic Methods in Applied Geophysics-THeory Volume I, chapter 4.