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**On the Convergence of Shifted Laplace Preconditioner
Combined with Multigrid Deflation**

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In this paper, a solver for the discretized time-harmonic wave equation in heterogeneous media is considered. The underlying equation governs wave propagations and scattering phenomena arising in many areas, e.g., in aeronautics, geophysics, and optical problems. In particular, we look for efficient iterative solution techniques for the Helmholtz equation discretized by finite difference discretizations. Since the number of grid points per wavelength should be sufficiently large for accurate solutions, for very high wavenumbers the discrete problem becomes extremely large, thus prohibiting the use of direct solvers. However, since the coefficient matrix is sparse, iterative solvers are an interesting alternative.

Starting point of our research is the multilevel Krylov method proposed in [1] which consists of a combination of the Shifted Laplace Preconditioner (SLP) given in [2] and a second level preconditioner based on Deflation [3]. It is well known that the coefficient matrix of the discretized Helmholtz equation has eigenvalues both in the positive and negative complex plane. Such a spectrum leads to a very slow convergence of Krylov type solvers. Combining the matrix with a Shifted Laplace Preconditioner leads to a spectrum which is enclosed in a circle in the positive complex plane. If no damping is included this circle passes through the origin. It appears that the resulting method is scalable in step size h but the number of iterations increases linearly with the increase of the wave number. The reason for this increase is that the number of eigenvalues close to zero increase considerably for large wave numbers. Motivated by the success of multigrid, where a smoother and coarse grid solver are combined, Erlangga and Nabben [1] use a combination of SLP and projection Deflation in a multilevel way. From their paper it appears that the number of iterations is much smaller and furthermore the method seems to be scalable also in the wave number. To gain insight in this method, we analyze the method proposed in more detail.

In this paper we perform a Rigorous Fourier Analysis of a two-level variant of

the multilevel Krylov method for the Helmholtz equation proposed in [1]. The distinct feature of the solver analyzed is the two-grid deflation of the Shifted Laplace Preconditioner at each step of an outer Krylov subspace acceleration. Our analysis of 1D and 2D models reveals two properties of the solver. The first is that with deflation, the solver performs better than without deflation. However, above a certain large wave number, the solver depends again linearly on the wave number. The reason is that the spectrum contains more small eigenvalues with increasing wave number. The second is that the Shifted Laplace Preconditioner can be made arbitrarily diagonally dominant by increasing the imaginary part of the shift without paying any penalty in the number of Krylov iterations. These properties are verified by numerical experiments on Helmholtz problems with heterogeneous media and may lead to faster methods to approximate the Shifted Laplace Preconditioner.

References

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