Nikolaos, N M Missirlis
Preconditioning augmented linear systems

Department of Informatics and Telecommunications
University of Athens
Panepistimiopolis
15784
Ilisia
Athens
Greece
nmis@di.uoa.gr
Maria Louka

Let $A \in \mathbb{R}^{m \times m}$ be a symmetric positive definite matrix and $B \in \mathbb{R}^{m \times n}$ be a matrix of full column rank, where $m \geq n$. Then, the augmented linear system is of the form

$$\tilde{A}u = b \tag{1}$$

where

$$\tilde{A} = \begin{pmatrix} A & B \\ -B^T & 0 \end{pmatrix}, \quad u = \begin{pmatrix} x \\ y \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ -b_2 \end{pmatrix} \tag{2}$$

with B^T be the transpose of the matrix B. When A and B are large and sparse, iterative methods for solving (1) are effective and more attractive than direct methods, because of storage requirements and preservation of sparsity.

The difficulty in applying iterative methods such as the successive overrelaxation (SOR) method [10] to system (1) is the singularity of the block diagonal part of the coefficient matrix. Some methods have been developed to overcome this difficulty such as the Uzawa and the Preconditioned Uzawa method [1]. In 2001, Golub et al. [4] generalized the Uzawa and the Preconditioned Uzawa method for the augmented system (1) by introducing an additional acceleration parameter and produced the SOR-like method. Other known methods, competitive and specially attractive for solving (1) are the preconditioned MINRES and GMRES methods [3]. In case where the matrix A is symmetric and positive definite, the Preconditioned Conjugate Gradient (PCG) method [5] is the most suitable. In 2003, Li, Evans and Zhang [6] proved that the PCG method is at least as accurate as the SOR-like method. In 2005, Bai et al. [2] proposed the Generalized SOR (GSOR) method and proved that it has the same rate of convergence but less complexity than the PCG method. Furthermore, the Generalized Modified Extrapolated SOR (GMESOR) method, a generalization of the GSOR method, was also proposed for further study. The latter is a generalization of the GSOR method as it uses one additional parameter.

In this paper, we study the Generalized Modified Extrapolated SOR (GME-SOR) method and the Generalized Modified Preconditioned Simultaneous Displacement (GMPSD) method. The latter is derived by using as preconditioning matrix the product of a lower and upper triangular part of $tildeA$ [7], [8]. In an attempt to generalize our theory, we introduce an additional parameter a in the structure of the matrix $\tilde{A}$, with the hope of increasing the rate of convergence for the aforementioned methods. Our starting point, for studying these methods, is the derivation of functional relations which relate the eigenvalues of their iteration matrices with those of the key matrix $J = Q^{-1}B^T A^{-1}B$. Assuming that the matrix Q is symmetric positive or negative definite matrix, the eigenvalues of the matrix J are real and either positive if the matrix Q is positive definite or negative if the matrix Q is negative definite. Under these assumptions we find sufficient conditions for the convergence of the aforementioned methods and determine the optimum values of their parameters. For the determination of the optimum values of the parameters in GMESOR and GMPSD methods we followed a similar approach to Varga [9] p. 110, for the optimum ω of SOR.

Bibliography

[1] K. Arrow, L. Hurwicz and H. Uzawa, Studies in Nonlinear Programming, Stanford University Press, Stanford, 1958.

[2] Z.-Z. Bai, B. N. Parlett and Z.-Q. Wang, On generalized succesive overrelaxation methods for augmented linear systems, Numer. Math. 102, (2005), 1-38.

[3] B. Fischer, R. Ramage, D. J. Silvester, A. J. Wathen, Minimum residual methods for augmented systems, BIT 38, (1998), 527-543.

[4] G. H. Golub, X. Wu and J.-Y Yuan, SOR-like methods for augmented systems, BIT 41, (2001), 71-85.

[5] M. R. Hestenes and E. Stiefel, Methods of Conjugate Gradients for Solving Linear Systems, Journal of Research of the National Bureau of Standards, vol. 49, No. 6, (1952), 409-436.

[6] C.-J. Li, Z. Li, D. J. Evans and T. Zhang, A note on an SOR-like method for augmented systems, IMA J. Numer. Anal. 23, (2003), 581-592.

[7] M. A. Louka, Iterative Methods for solving linear systems, Ph.D thesis (in Greek), 2011.

[8] M. A. Louka, N. M. Missirlis and F. I. Tzaferis, Is modified PSD equivalent to modified SOR for two-cyclic matrices? Lin. Alg. and its Appl., Vol. 432, No. 11, (2010), 2798-2815.

[9] R. S. Varga, Matrix Iterative Analysis, Prentice-Hall, Inc. Englewood Cliffs, N.J., 1962.

[10] D. M. Young, Iterative Solution of Large Linear Systems, Academic Press, New York, 1971.