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**Extending GMRES to Nonlinear Optimization, with  
Application to Tensor Optimization**

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GMRES is a powerful method to accelerate convergence of simple iterative solvers for linear systems, like Jacobi or Gauss-Seidel (and other preconditioning processes). We extend the concept of preconditioned GMRES to the general class of nonlinear optimization problems, proposing a general mechanism to accelerate simple iterative methods for nonlinear optimization problems that works in a genuinely nonlinear way.

Each iteration of the method consists of three steps. In the first step, a tentative new iterate is generated by a stand-alone one-step process (which can be nonlinear). In the second step, an accelerated iterate is generated by a nonlinear GMRES approach, recombining previous iterates in an optimal way, and essentially using the stand-alone one-step process as a preconditioner. In particular, the nonlinear extension of GMRES is used that was proposed by Washio and Oosterlee in [ETNA Vol. 15 (2003), pp. 165-185] for nonlinear partial differential equation problems (which is itself related to other existing acceleration methods for nonlinear equation systems). In the third step, a line search is performed for globalization.

The resulting nonlinear GMRES (N-GMRES) optimization algorithm is applied to a set of standard test problems for nonlinear optimization, using steepest descent as a universally applicable preconditioner (corresponding to the identity preconditioner in GMRES for linear systems). Performance of steepest-descent preconditioned N-GMRES is shown to be competitive with standard nonlinear conjugate gradient (N-CG) and limited-memory Broyden-Fletcher-Goldfarb-Shanno (L-BFGS) optimization methods for the model problems considered. A simple global convergence proof is provided for the N-GMRES optimization algorithm with steepest descent preconditioner (with line search), under mild standard conditions on the objective function and the line search processes.

Finally, the performance of the N-GMRES optimization algorithm is also illustrated for the problem of computing a canonical rank-R tensor approximation

that has minimal distance to a given tensor in the Frobenius norm, where the canonical rank- $R$  tensor consists of the sum of  $R$  rank-one tensors. For this application, we use alternating least-squares (ALS) minimization as the nonlinear preconditioning process, and we find that ALS-preconditioned N-GMRES is significantly faster than N-CG and L-BFGS for this difficult nonlinear optimization problem. This illustrates how the N-GMRES optimization framework allows for the use of powerful problem-dependent nonlinear preconditioners, and we anticipate that this strategy may lead to efficient numerical methods for a variety of nonlinear optimization problems.